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The Use of Geometric Algebra in the Analysis of Non-sinusoidal Networks and
the Construction of a Unified Power Theory for Single Phase Systems - A Paradigm Shift

by

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Abstract

The electrical engineering scientific community since 1892 is seeking a power theory for interpreting the power flow within electric networks under non-sinusoidal conditions. The proliferation of power electronic devices in electrical systems provides added motivation to find such a theory. Some examples of the effort regarding power definitions and measurements under non-sinusoidal conditions include four international workshops, an IEEE working group and a biannual international conference.

Although many power theories have been proposed regarding non-sinusoidal operation, an adequate solution is yet to be found. In contrast to previous investigations, it is suggested here that the framework based on complex number representations in non-sinusoidal circuit analysis may in fact hamper energy flow analysis. Thus, a new circuit analysis approach is developed using geometric algebra. In a new domain – coined as the GN domain – multivectors describe circuit and power quantities, circuit quantities obey Kirchhoff’s circuit laws, it is possible to apply the superposition principle and a better sense of the flow of currents and powers in the examined circuits is shown.

The power multivector results from the geometric product of the voltage and current multivectors. The power multivector allows a decomposition that accounts for the total active and non-active power, involves the well-known power terms of the sinusoidal case – reactive and active average power – and two new terms: degrading power and reactive power due to harmonic interactions. Also, the power multivector satisfies both: the principle of conservation of energy and the balance principle of reactive power. The proposed GN domain power theory is able to: 1) reveal flaws in other power theories, 2) undermines the concept that the number of reactive elements required to achieve a near unity power factor is dependent on the number of harmonics in the excitation source, and 3) shows that the traditional non-sinusoidal apparent power definition needs to be revised. Presently, no power theory has these features all together.
Acknowledgements

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Deysy Londoño-Monsalve deserves a special note of appreciation for reviewing the manuscripts of all publications and discussing many adventurous ideas. Her constant support is the most important reason that this dissertation has been completed. I am forever grateful to you my dear Deysy.

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The author also expresses his deepest gratitude to Dr. Alexander Petroianu, emeritus professor at University of Cape Town – South Africa, for awaking my interest and curiosity on this unresolved, century-old problem and the field of geometric algebra.

Last but certainly not least, the author recognizes the invaluable support of Róbinson Castro-Puche, emeritus professor at University of Córdoba – Colombia, for teaching me, with such an outstanding pedagogical treatment, the field of geometric algebra from the abstract algebra perspective. Without his support I would have not been able to develop this novel circuit analysis approach and its associated power theory, nor would I have developed the mathematical skills to perceive the weakness of the traditional tool used today in circuit analysis. Gracias Papi!
Dedication

To EL-ROY – the God of Abraham, Jacob and Isaac and now the Lord my God in whom I will trust.

To Deysy – My lovely wife, Valery – My immensely loved princess and Daniel – My champion and hero, with deepest love and gratitude.

To Róbinson – My father, Olga – My beloved mother and source of motivation, Tania – My beloved sister, Jaime – My source of inspiration and Glenna – My source of admiration, with deepest love and gratitude to them all

To Jaime Blandon-Diaz – My best boss ever and Armando Potes-Gutiérrez – The best teacher of physics I have known

To my professors at Universidad Nacional de Colombia – *The creators of the engineer I am*, Universidad de Antioquia, University of Manitoba and University of Calgary

To every person, too many to name them all, that have accompanied me on my journey through life and have helped me become a better person
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<td>$CN_{r(ps)}$</td>
<td>Reactive power due to voltage/current phase-shift</td>
</tr>
<tr>
<td>$CN_{r(hi)}$</td>
<td>Reactive power due to harmonic interactions</td>
</tr>
<tr>
<td>$CN_d$</td>
<td>Degrading power</td>
</tr>
<tr>
<td>$CN$</td>
<td>Non-active power multivector</td>
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<tr>
<td>$DB$</td>
<td>Budeanu’s distortion power</td>
</tr>
<tr>
<td>$DH$</td>
<td>Harmonic distortion power</td>
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<tr>
<td>$D_I$</td>
<td>Current distortion power</td>
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<td>$D_S$</td>
<td>Czarnecki’s scattered power</td>
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<tr>
<td>$DV$</td>
<td>Voltage distortion power</td>
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<tr>
<td>$I$</td>
<td>Current multivector</td>
</tr>
<tr>
<td>$Im$</td>
<td>Imaginary part of a complex number</td>
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<td>$i_a$</td>
<td>Czarnecki’s active current</td>
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<tr>
<td>$i_r$</td>
<td>Czarnecki’s reactive current</td>
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<tr>
<td>$i_s$</td>
<td>Czarnecki’s scattered current</td>
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<td>$M$</td>
<td>Power multivector</td>
</tr>
<tr>
<td>$N$</td>
<td>Non-active power</td>
</tr>
<tr>
<td>$P$</td>
<td>Active average power and scalar part of $M$</td>
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<tr>
<td>$P_1$</td>
<td>Fundamental active average power</td>
</tr>
<tr>
<td>$P_H$</td>
<td>Non-fundamental active average power</td>
</tr>
<tr>
<td>$PoCoE$</td>
<td>Principle of Conservation of Energy</td>
</tr>
<tr>
<td>$p_f$</td>
<td>Power factor</td>
</tr>
<tr>
<td>$\langle p(t) \rangle$</td>
<td>Average of the instantaneous power</td>
</tr>
<tr>
<td>$Q$</td>
<td>Reactive power in sinusoidal systems</td>
</tr>
<tr>
<td>$Q_1$</td>
<td>Fundamental reactive power</td>
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<tr>
<td>$Q_B$</td>
<td>Budeanu’s reactive power</td>
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<td>$Q_F$</td>
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<tr>
<td>$Q_r$</td>
<td>Czarnecki’s reactive power</td>
</tr>
<tr>
<td>$Re$</td>
<td>Real part of a complex number</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The set of real numbers</td>
</tr>
<tr>
<td>$S$</td>
<td>Complex power and its Magnitude</td>
</tr>
<tr>
<td>$|S|$</td>
<td>Magnitude of complex power</td>
</tr>
<tr>
<td>$S_N$</td>
<td>Non-fundamental apparent power</td>
</tr>
<tr>
<td>$S_1$</td>
<td>Fundamental apparent power</td>
</tr>
<tr>
<td>$V$</td>
<td>Voltage multivector</td>
</tr>
<tr>
<td>$VI$</td>
<td>Geometric product of multivectors $V$ and $I$</td>
</tr>
<tr>
<td>$V \cdot I$</td>
<td>Internal product of multivectors $V$ and $I$</td>
</tr>
<tr>
<td>$V \wedge I$</td>
<td>External product of multivectors $V$ and $I$</td>
</tr>
<tr>
<td>$v(t)$ and $i(t)$</td>
<td>Voltage and current signal in the time-domain</td>
</tr>
<tr>
<td>$\sigma_n$</td>
<td>The $n^{th}$ unit 1-vector</td>
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<tr>
<td>$\sigma_1 \sigma_2$</td>
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<td>The $n$-vector representing the $n^{th}$ harmonic of a cosine wave and sine wave in the $\mathbb{G}_{3N}$ algebra</td>
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Chapter One: Introduction

1.1 An Unresolved Century-Old Challenge

The challenge of developing a unified power theory for electric networks suitable for sinusoidal and non-sinusoidal conditions can be traced back to 1892 when Steinmetz showed that in circuits with electric arcs the apparent power is greater than the active power [1]. Ever since that time power theories have dealt with the challenge of responding the following two questions [2]:

1) Why does the supply source usually require a higher apparent power than the active power to feed an electric load?
2) How is it possible to maintain the active power required by the load while reducing the apparent power of the supply source?

Many power theories have been developed over the years; moreover, several articles dealing with the nature of power are published each year and therefore the number of publications dealing with power theory is significant [3]. The first formal power theory, developed by Budeanu [4], appeared in 1927 and is the most widely accepted power theory today by electrical engineers. Nevertheless, Budeanu’s power theory was criticized as early as 1930 by Fryze [5] who developed a power theory using the time domain. Although both theories fail to properly describe the power phenomena in non-sinusoidal conditions they serve more or less as the base for many other power theories. On the other hand, the two most recent power theories of major impact, yet with no general acceptance today are:

a) The instantaneous power theory proposed in 1983 by Akagi, Nabae and Kanazawa [6], [7], and

Despite the research on power theories since 1920, all power theories make use of only two domains, the time-domain and the frequency-domain.

1.2 The Ever-Increasing Need for a Unified Theory of Power

Power electronic devices have brought numerous benefits to power systems and to industrial applications, e.g. the development of High Voltage Direct Current (HVDC) systems, Flexible AC Transmission Systems (FACTS) and Variable Speed Drives (VSD), among much other advancement. Unfortunately, the proliferation of power electronic devices in electric networks has also increased the presence of distorted voltages and currents [9]. This increased presence of distorted voltages and currents has brought in turn the additional challenge of understanding how energy flows in non-sinusoidal conditions, i.e. when the voltage and the current are distorted. Although the existing standards attempt to reduce the distortion of voltages and currents, this approach is insufficient as the penetration of power electronic devices is projected to increase dramatically in the following decades. Thus, the proliferation of power electronic devices in the electric power system has manifested the necessity for a theory of power capable of more completely explaining the phenomena of power in non-sinusoidal situations and still be compatible with the existing power theory used in sinusoidal conditions.

So far, and for over a century, the efforts on developing a power theory have concentrated on how to decompose the traditional definition of apparent power – where apparent power is a quantity defined as the product of two magnitudes related to voltage and current. In contrast, this dissertation takes a divergent approach. Thus, for the first time, the frequency-domain framework is challenged and the geometric algebra framework, referred to here as the $G_N$ domain, is used to develop a new circuit analysis approach and to develop a conservative power theory suitable for sinusoidal and non-
sinusoidal conditions using the new circuit analysis approach. The principle of conservation of energy, hereinafter referred to as the PoCoE, entails the time integral of the instantaneous volt-ampere product as Equation (1.1) depicts [10].

\[ \int_{-\infty}^{\infty} \left( \sum_g^{\alpha} \sum_k^{\beta} (vi)_{kg} \right) dt = 0 \]  

(1.1)

In Equation (1.1) the term \((vi)_{kg}\) is the instantaneous volt-amperes in the \(k^{th}\) parallel sub-branch on the \(g^{th}\) series branch [10]. It follows from Equation (1.1) that if \(\sum_g^{\alpha} \sum_k^{\beta} (vi)_{kg} = 0\) then \(\int_{-\infty}^{\infty} (\sum_g^{\alpha} \sum_k^{\beta} (vi)_{kg}) dt = 0\). The implication of \(\sum_g^{\alpha} \sum_k^{\beta} (vi)_{kg} = 0\) given as Rule 1 in [10] is that: “In any electric circuit, at any instant, the instantaneous rate of energy transfer (instantaneous volt-amperes or instantaneous power) at the input terminals is equal to the sum of the instantaneous rate of energy transfer in the various loads components.” From a rigorous mathematical perspective in turns out that Rule 1 is actually a corollary of the PoCoE; therefore, whenever this corollary holds true, the principle of conservation of energy is inevitably fulfilled. Since the term \(\sum_g^{\alpha} \sum_k^{\beta} (vi)_{kg} = 0\) requires that the total instantaneous volt-amperes at the input terminal must be equal to the sum of the instantaneous volt-amperes at each load component; throughout the thesis, the fulfillment of this corollary guarantees the fulfillment of the PoCoE.

1.3 Research’s Objectives

The work in this dissertation is based on the premise that the present state of affairs of power theories is a consequence of the tool used to analyze circuits. Therefore the dissertation presents a new circuit analysis approach that serves as the platform to develop a unified power theory that is conservative and suitable for both sinusoidal and non-sinusoidal conditions. Both, the proposed new circuit analysis approach and the proposed new single phase power theory use the geometric algebra framework. Thus, the dissertation’s objectives are:
1) To examine, from a rigorous mathematical perspective, the framework of past and present circuit analysis techniques and determine the impact of the mathematical weakness of this framework on past and present power theories.

2) To provide a new circuits analysis approach based on the framework of geometric algebra and compare the new approach with the frequency-domain framework developed by Steinmetz and others,

3) To construct a conservative power theory for single-phase systems where the source can be sinusoidal or non-sinusoidal and the load is either of a harmonic generation type or composed by linear circuit elements, and

4) To contrast the power theory developed in this dissertation with four well-established power theories, i.e., Budeanu’s, Fryze’s, The IEEE standard 1459-2010 and Czarnecki’s CPC power theory.

1.4 Organization of the Dissertation

This dissertation is divided into six chapters and although the dissertation’s main contributions are given in Chapters 4 and 5, Chapter 3 reveals the key elements to understand the root cause of the challenge.

Chapter 1 provides an overview of the challenge of developing a power theory that explains properly the phenomena of power in electric networks when the voltages and currents are non-sinusoidal. The chapter also highlights the reasons for the necessity of a new power theory.

Chapter 2 provides a literature review of the major work done by C. Budeanu, Fryze, the team work that yielded the IEEE standard 1459-2010 and Czarnecki’s CPC power theory. The chapter ends with a short list of the numerous approaches scholars and scientists have taken to solve a century-old, unresolved problem of constructing a power theory suitable for sinusoidal and non-sinusoidal conditions.
Chapter 3 examines three aspects of the challenge of developing a non-sinusoidal power theory: the limitation of Steinmetz’s non-sinusoidal circuit analysis framework; the mathematical weaknesses in the present definition of some power and circuit quantities; and the unsuitability of the present definition of apparent power for interpreting the physical phenomena of power in non-sinusoidal conditions.

Chapter 4 deals with basic concepts of geometric algebra, the new circuit analysis approach using geometric algebra and the development of the proposed conservative power theory.

Chapter 5 uses Kuhn’s five criteria and the principle of conservation of energy to examine the power theory presented in this dissertation and the four power theories described in chapter 2.

Chapter 6 provides conclusions and summarizes the main contributions of this dissertation. It also suggests possible directions for future work on power flow analysis using geometric algebra.
2.1 Classical Power Theory in Sinusoidal Conditions

In the time-domain, the concept of power entails the product of the voltage signal \( v(t) \) e.g. Equation (2.1) and a current signal \( i(t) \) e.g. Equation (2.2). Thus, the concept of power results from the definition \( p(t) = v(t)i(t) \) as given by Equation (2.3). The concept of power is of significant importance in electrical engineering for many reasons, e.g., billing purposes, optimal operation of power systems, investment cost assessments and efficient use of energy, among many others. The quantities \( V \) and \( I \) in Equations (2.1) to (2.3) are RMS values and the angle \( \pm \varphi \) represents the phase-shift between the voltage and the current signals. Thus,

\[
v(t) = \sqrt{2}V \cos(\omega t + \alpha) \tag{2.1}
\]

\[
i(t) = \sqrt{2}I \cos(\omega t + \alpha \pm \varphi) \tag{2.2}
\]

\[
p(t) = v(t)i(t) = 2VI \cos(\omega t + \alpha) \cos(\omega t + \alpha \pm \varphi) \tag{2.3}
\]

Equation (2.3) indicates instantaneous power and is usually rewritten in the form of Equation (2.4) to define other power quantities. Thus,

\[
p(t) = \frac{VI \cos \varphi}{p} + \frac{VI \cos \varphi \cos(2\omega t + 2\alpha)}{p} \mp \frac{VI \sin \varphi \sin(2\omega t + 2\alpha)}{Q} \tag{2.4}
\]

The average value of the power signal \( p(t) \), denoted by \( \langle p(t) \rangle \) and assessed over the period of time \( T_0 \) is given by,
In Equation (2.5) \( T_0 \) is often set to the period of voltage and current signals. Since the average of \( p(t) \) yields the value \( VI \cos \varphi \), which is the first term in (2.4), then the active average power \( P \) is defined as the scalar component of the instantaneous power; thus, \( \langle p(t) \rangle = P = VI \cos \varphi \), is the standard definition of the active average power. There is no controversy about this definition in the literature. The IEEE standard 270-2006 defines the active power \( P \) as the rate by which work is done or energy is transferred and it emphasizes that it must be differentiated from the quantities reactive power and apparent power. The amplitude of the third term in Equation (2.4) i.e. \( VI \sin \phi \sin(2\omega t + 2\alpha) \) is \( VI \sin \phi = Q \); therefore, the concept of reactive power is defined as the amplitude of one of the two oscillatory terms in the instantaneous power equation. This is in contrast to the definition of the active power, which is based on the result of a mathematical operation in the time-domain. The IEEE standard 270-2006, defines reactive power as a quantity that describes power that flows back and forth in an ac circuit without being consumed, and adds that in sinusoidal situations this quantity can be estimated also by the following definition,

\[
Q = EI_q
\]  
(2.6)

In Equation (2.6) \( E \) represents the RMS value of the voltage signal denoted in this dissertation as \( V \), i.e. \( E = V \) and \( I_q \) represents the in-quadrature component of the current \( I \) in relation to the voltage \( V \). There is also an in-phase component of the current \( I \) in relation to the voltage \( V \) denoted by \( I_p \). This in-phase component allows the subsequent definition of active power,

\[
P = EI_p
\]  
(2.7)
Inspection of Equation (2.4) shows that the ac term $VI \cos \varphi \cos(2\omega t + 2\alpha)$ of the instantaneous power equation is completely ignored; more importantly yet, no technical or theoretical reason or mathematical support is given for such dismissal. However, Ghassemi [11] considers this term must be included in a thorough analysis as it affects energy oscillation between source and load. In an effort to overcome this imperfection, the IEEE Non-Sinusoidal Situation Working Group states in the IEEE standard 1459-2010 that the term $P \cos(2\omega t)$ results from the second term of the equation $p(t) = VI \cos \varphi + VI \cos \varphi \cos(2\omega t + 2\alpha)$ and is called intrinsic power. Nevertheless, no further explanation is given beside the fact that it is an oscillatory component, that it is always present in transportation of energy and that it causes no losses.

In addition to the notion of active and reactive power, there are two other concepts in the classical power theory under sinusoidal conditions; these are the complex power denoted by $S$ and the apparent power, usually represented by $||S||$ or simply $S$. Interpretation of Equation (2.4) shows that the terms $P$ and $Q$ represent the power terms resulting from two different phenomena. The first phenomenon, which is characterized by the active power $P$ results when the voltage and the current are in-phase; while the second phenomenon, which is characterized by the reactive power $Q$, results when the voltage and the current are in-quadrature. The IEEE standard 270-2006, provides the following two definitions of apparent power,

$$||S||^2 = P^2 + Q^2$$
$$S = EI = VI$$  \hspace{1cm} (2.8a) \hspace{1cm} (2.8b)

In addition to the above definition, the IEEE standard 1459-2010 adds that for single phase loads, the apparent power can be interpreted as the maximum active power that can be transmitted through the same line while keeping constant both, the load’s rms voltage and the supplying current, (constant line losses).
As the current can be decomposed into the in-phase component and the in-quadrature component as in Equation (2.9) below, then it is possible the use of a two-dimensional framework. Thus,

\[
\begin{align*}
    I_p &= I \cos \varphi \\
    I_q &= I \sin \varphi \\
\end{align*}
\]

Since

\[
\begin{align*}
    I_p &= I \cos \varphi \\
    I_q &= I \sin \varphi \\
    \text{then} \\
    I^2_p + I^2_q &= I^2 \cos^2 \varphi + I^2 \sin^2 \varphi = I^2 \\
    I &= \sqrt{I^2_p + I^2_q}
\end{align*}
\]

Note that the algebra of complex numbers with its associated complex plane emerges immediately as the ideal framework choice. Thus, the current can be decomposed in the complex plane as \( I = I_p + jI_q \), where \( ||I|| = \sqrt{I^2_p + I^2_q} \). Consequently, the quantities \( P, Q \) and \( S \) can be rewritten as,

\[
P = EI_p \tag{2.10}
\]

\[
jQ = jEI_q \tag{2.11}
\]

\[
S = V I \cos \varphi + jV I \sin \varphi = V I^* = P + jQ \tag{2.12a}
\]

\[
||S|| = ||V|| ||I^*|| = \sqrt{P^2 + Q^2} \tag{2.12b}
\]

In Equation (2.12) \( S, V \) and \( I^* \) denote complex quantities while \( P \) and \( Q \) denote scalar quantities. In addition, \( V \) and \( I^* \) denote phasor quantities and \( I^* \) represents the complex conjugate of the current phasor \( I \). Chapter Three explores in more detail the concept of phasor, the limitations of Steinmetz’s non-sinusoidal circuit analysis framework and the consequences of these limitations. For now, it is enough to say that the phasor representation of Equations (2.1) and (2.2) is given by \( V = ||V|| \angle \alpha, \ I = ||I|| \angle (\alpha \pm \varphi) \) and \( I^* = ||I|| \angle - (\alpha \pm \varphi) \). The complex power \( S \) has no physical significance or interpretation as this quantity appears more as a lucky coincidence [12]; consequently,
the apparent power $||S||$ does not have physical interpretation either [13], regardless of whether the waveform in sinusoidal or not. However, the apparent power concept is often used in practice as a meaningful measure for the rating of a particular device, assuming the nominal voltage of the equipment is given or known.

There is yet one more concept in sinusoidal power theory, the power factor concept. The IEEE standard 1459-2010 as well as the IEEE standard 270-2006, provides the following definition for the power factor concept,

$$pf = \frac{P}{S}$$ (2.13)

In Equation (2.13) $P$ denotes the active average power while $S$ denotes the apparent power. The IEEE standard 1459-2010 adds that provided the supplying current is kept constant (line losses constant), the power factor can be interpreted as the ratio between the energy transmitted to the load over the maximum energy that could be transmitted. Notice that “the maximum energy that could be transmitted,” which is done with the aid of a transmission line, implies a physical interpretation. That is, according to the IEEE standard 1459-2010 the concept of apparent power has a physical interpretation. However, authorities such as Czarnecki [13] and Ilić [12] among others, reject the idea of a physical interpretation for the apparent power concept, even in single-phase sinusoidal situations. The IEEE standard 1459-2010 states also that the ratio $PF = \frac{P}{S}$ is a factor that indicates the degree of utilization of the line. Such statement indicates a physical interpretation; however, how can a quantity have physical interpretation when its definition involves another quantity that has no physical interpretation? Either it is accepted that the apparent power concept has physical significance, which will lead the power factor concept to claim its physical significance, or it is accepted that the apparent power does not have a physical interpretation which will yield a lack of physical interpretation for the power factor concept.
In [3] Czarnecki states that it is a trivial expectation for power theories to obey the rules of mathematics and physics; however, what mathematical structure, (group, ring, field), supports the addition of \(P\), which results from the mean value of the instantaneous power signal and \(Q\), which results from the amplitude of an oscillatory component of the same instantaneous power signal? According to Czarnecki, the latter is only a pseudo-problem; however, are pseudo-problems above the mathematical rigor or beyond the rules of physics? What is the definition of a pseudo-problem? How the scientific method determines when a pseudo-problem is acceptable and when it is not acceptable? What is the rationale applied by the scientific community to accept or reject a theory that involves a pseudo-problem?

2.2 Power Theory Proposed by C. Budeanu

Constantine I. Budeanu’s power theory is detailed in his 1927 book titled *Reactive and Fictive Powers* [4]. His theory, developed in the frequency-domain, can be explained as follows. Let \(I_h\) be the harmonic current response to the excitation of the harmonic voltage \(V_h\) and \(\varphi_h\) the phase angle between the voltage and the current signals. Thus,

\[
\begin{align*}
V_h &= V_h \cos \alpha + i I_h \sin \alpha &\rightarrow & \text{Excitation signal} \\
I_h &= I_h \cos \varphi_h + i I_h \sin \varphi_h &\rightarrow & \text{Response}
\end{align*}
\] (2.14)

In Equation (2.14) all quantities belong to the frequency-domain. The apparent power squared is given by

\[
S^2 = V^2 I^2 = \sum_{h=1}^{n} V_h^2 \left[\sum_{h=1}^{n} (I_h \cos \varphi_h)^2\right] + \sum_{h=1}^{n} V_h^2 \left[\sum_{h=1}^{n} (I_h \sin \varphi_h)^2\right] \tag{2.15}
\]

Grouping terms appropriately yields
The apparent power can now be separated into three terms, namely, active average power $P$, reactive power $Q_B$ and distortion power $D_B = \sqrt{D}$. As almost every author of a power theory uses the letter $Q$ to denote the reactive power; the sub index $B$ is used to denote Budeanu’s reactive power definition. Thus,

\[
P = \sum_{h=1}^{n} V_h I_h \cos \varphi_h
\]  

\[
Q_B = \sum_{h=1}^{n} V_h I_h \sin \varphi_h
\]  

\[
D_B = \sqrt{\sum_{p=1}^{n-1} \sum_{q=p+1}^{n} \left[ (V_p I_q)^2 + (V_q I_p)^2 - 2V_p V_q I_p I_q \cos(\varphi_p - \varphi_q) \right]}
\]

\[
S^2 = P^2 + Q_B^2 + D_B^2
\]

When describing Budeanu’s power theory, it is a common approach to start by providing the definition of the active average power $P$ and reactive power $Q_B$ followed by the definition of the distortion power $D_B$ based on the apparent power; thus, the distortion power is defined as $D_B = \sqrt{S^2 - P^2 - Q_B^2}$. This approach is misleading from the historical perspective and more importantly, from the perspective of the actual development of the theory. Notice that in the approach described above by Equations (2.17) to (2.20) the definition of all the power terms is a result. In contrast, the common approach places the results as the starting point, it denies the ingenuity of the mathematical process and leaves the erroneous impression that the definition of the distortion power emerges from the necessity to reestablish the balance in the power equation. This was not the case, the realization of an additional power term besides the active average power and the reactive power resulted from a mathematical interpretation. Budeanu was the first scientist to
propose a three-dimensional representation of the apparent power and consequently the first one to realize the appearance of a new power quantity produced by a different phenomenon.

2.3 Power Theory Proposed by S. Fryze

Stanislaw Fryze presented his power theory in 1932 [5]. His power theory has had a significant impact until today. The FBD power theory developed by Depenbrock [14] and the Current’s Physical Components (CPC) power theory developed by Czarnecki [8] are heavily influenced by Fryze’s approach and constitute prominent examples of the scope of his influence. Indeed, the acronym FBD stands for Fryze-Buchholz-Depenbrock.

Fryze’s approach extends the concept of active and reactive currents in the time-domain in sinusoidal situations to the non-sinusoidal case; his theory, developed in the time-domain, is better understood when the sinusoidal case is examined first. Thus, let 

\[ i(t) = I \sqrt{2}\sin(\omega t - \varphi) \]

be the response to the voltage excitation \( v(t) = V \sqrt{2}\sin(\omega t) \), where \( V \) and \( I \) denote the RMS values of the voltage and the current, and \( \varphi \) denotes the phase angle between the two time-domain signals. Thus,

\[
\begin{align*}
V_h &= V \sqrt{2}\sin(\omega t) \quad \rightarrow \quad \text{Excitation signal} \\
I_h &= I \sqrt{2}\sin(\omega t - \varphi) \quad \rightarrow \quad \text{Response}
\end{align*}
\]

(2.21a) \hspace{1cm} (2.21b)

Notice that the current signal can be decomposed in the time-domain as Equation (2.22) below shows. Thus,

\[
I_h = I \sqrt{2}\sin(\omega t - \varphi) = I \sqrt{2}[\sin(\omega t)\cos(\varphi) - \sin(\varphi)\cos(\omega t)]
\]

(2.22)

Notice that the first component of Equation (2.22) is in phase with the voltage while the second component is in quadrature with voltage. Therefore, the active current can be
described by the component in-phase with the voltage while the reactive current can be described by the component in-quadrature with the voltage. Thus,

\[
\begin{align*}
\begin{cases}
i_a(t) = I \sqrt{2} \cos(\varphi) \sin(\omega t) = I_a \sin(\omega t) \\
\text{where} \\
I_a = I \sqrt{2} \cos(\varphi)
\end{cases} & \quad (2.23a) \\
\begin{cases}
i_r(t) = -I \sqrt{2} \sin(\varphi) \cos(\omega t) = -I_r \cos(\omega t) \\
\text{where} \\
I_r = I \sqrt{2} \sin(\varphi)
\end{cases} & \quad (2.23b)
\end{align*}
\]

Now, since there is a well-established relation between the conductance \(G\) and the active current \(I_a\) and between \(I_a\) and the active average power \(P\), then it is possible to establish a relation between the conductance \(G\) and the active average power \(P\). Thus,

\[
G = \frac{I_a}{V} = \frac{I_a V}{V V} = \frac{P}{V^2}
\]

(2.25)

Following the same process it is possible to drive a relation between the susceptance \(B\) and the reactive power \(Q\), then:

\[
B = \frac{I_r}{V} = \frac{I_r V}{V V} = \frac{Q}{V^2}
\]

(2.26)

The power terms are then given by

\[
P = V I_a
\]

(2.27)

\[
Q = V I_r
\]

(2.28)

\[
S = V \sqrt{I_a^2 + I_r^2}
\]

(2.29)
Since in the sinusoidal case the active current is in phase with the voltage then it appears logical to extrapolate this concept to non-sinusoidal situations. Thus, Fryze defined the active current in non-sinusoidal situations as the current that is in phase with the non-sinusoidal voltage. However, this requires defining first an equivalent conductance which can be done by applying a modified version of Equation (2.25) that is suitable for non-sinusoidal conditions. Thus,

\[ v(t) = \sum_{h=1}^{n} V_h \sqrt{2} \sin(h \omega t + \varphi_h) \]  

(2.30)

In Equation (2.30) \( V_h \) is the RMS value of the voltage’s \( h^{th} \) harmonic, while \( \varphi_h \) is the voltage’s phase angle of the \( h^{th} \) harmonic. The equivalent conductance can be defined as,

\[
\begin{aligned}
    G &= \frac{P}{V^2} \\
    \text{where,} \\
    P &= \sum_{h=1}^{n} V_h I_h \cos \varphi_h \\
    \|v(t)\| &= \sqrt{\frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} v^2(t) \, dt} = \sqrt{\sum_{h=1}^{n} V_h^2} = V
\end{aligned}
\]

(2.31a)

(2.31b)

(2.31c)

\( V_h^2 \) is the RMS squared of the voltage’s \( h^{th} \) harmonic

The instantaneous active current can be defined as:

\[
\begin{aligned}
    (i_a(t)) &= \frac{G v(t)}{\sqrt{2}} = \sum_{h=1}^{n} I_{ah} \sqrt{2} \sin(h \omega t + \varphi_h) = \sum_{h=1}^{n} i_{ah}(t) \\
    \text{where,} \\
    I_{ah} &= GV_h \\
    i_{ah}(t) &= I_{ah} \sqrt{2} \sin(h \omega t + \varphi_h) \\
    \|i_a(t)\| &= \sqrt{\sum_{h=1}^{n} I_{ah}^2} = I_a
\end{aligned}
\]

(2.32a)

(2.32b)

(2.32c)

(2.32d)

Since in the sinusoidal case the reactive current is the current that is not active; then Fryze came to the conclusion that what is not active must be reactive. Therefore, Fryze defines the reactive current as the subtraction between the total instantaneous current and the instantaneous active current. Thus,
Using Equations (2.30) through (2.33) leads to the definitions of the power quantities in Fryze’s model; which are indeed extensions of the sinusoidal case. Thus,

\[\begin{align*}
  i_{RF}(t) &= i(t) - i_a(t) \\
  \left\| i_{RF}(t) \right\| &= \sqrt{\frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} [i^2(t) - i_a^2(t)]dt} = \sqrt{\sum_{h=1}^{\infty} I_{h-rF}^2} = I_{RF} = I \\
  \text{where} & \\
  I_{h-rF}^2 &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} [i^2(t) - i_a^2(t)]dt \\
  I &= \left\| i(t) \right\|
\end{align*}\] (2.34)

The electric power quantities for single-phase non-sinusoidal networks are established in section 3.1.2 of the IEEE Standard 1459-2010 [15]. The IEEE Standard uses the frequency-domain; thus, all the equations and quantities belong to the frequency-domain. In the Standard, voltages and the currents are divided in two components: the fundamental frequency components denoted by the sub index 1 and the non-fundamental components denoted by the sub index \(H\). The direct voltage and the direct current terms \((v_0, i_0)\) are included as part of the non-fundamental components. Therefore, in the frequency-domain the RMS value squared yields,

\[V^2 = V_1^2 + V_H^2\] (2.37)
\[ I^2 = I_1^2 + I_H^2 \]  \hspace{1cm} (2.38)

where

\[ V_H^2 = V_0^2 + \sum_{h \neq 1} V_h^2 \]  \hspace{1cm} (2.39)

\[ I_H^2 = I_0^2 + \sum_{h \neq 1} I_h^2 \]  \hspace{1cm} (2.40)

The active power \( P \) is also divided into fundamental frequency component and the non-fundamental components.

\[ P = P_1 + P_H \]  \hspace{1cm} (2.41)

In Equation (2.41) the terms \( P_1 \) and \( P_H \) are defined in Equations (2.42) and (2.43). Thus,

\[ P_1 = V_1 I_1 \cos \theta_1 \]  \hspace{1cm} (2.42)

\[ P_H = P - P_1 \]  \hspace{1cm} (2.43)

The apparent power \( S \) is divided into the fundamental apparent power \( S_1 \) and the non-fundamental apparent power \( S_N \). The term \( S_1 \) is formed by the fundamental active power \( P_1 \) and the fundamental reactive power \( Q_1 \). The term \( S_N \) is formed by the current distortion power \( D_I \), the voltage distortion power \( D_V \), and the harmonic apparent power \( S_H \). The term \( S_H \) is composed by the harmonic distortion power \( D_H \) and the non-fundamental active power \( P_H \). The Standard defines one more power term called Nonactive power \( N \). This term lumps the fundamental and the non-fundamental non-active components. Equations (2.44) to (2.55) below provide the definitions of all these quantities.
\[ S = VI \] (2.44)

\[ S^2 = (VI)^2 = (V_1I_1)^2 + (V_HI_H)^2 + (V_HI_1)^2 + (V_HI_H)^2 \] (2.45)

\[ S^2 = S^2_I + S^2_N \] (2.46)

\[ S_1 = V_1I_1 \] (2.47)

\[ S^2_I = P^2_I + Q^2_I \] (2.48)

\[ Q_1 = V_1I_1 \sin \theta_1 \] (2.49)

\[ S_N = \sqrt{S^2 - S^2_I} = D^2_I + D^2_V + S^2_H \] (2.50)

\[ D_I = V_1I_H \] (2.51)

\[ D_V = V_HI_1 \] (2.52)

\[ S_H = V_HI_H = \sqrt{P^2_H + D^2_H} \] (2.53)

\[ D_H = \sqrt{S^2_H - P^2_H} \] (2.54)

\[ N = \sqrt{S^2 - P^2} \] (2.55)

The Standard stresses that: “the non-active power \( N \) is not reactive power. Only when the waveforms are perfectly sinusoidal, \( N=Q_I=Q \)” [15]. The ratio of fundamental active power to fundamental apparent power defines the fundamental power factor \( PF_1 \) given by Equation (2.56) while the ratio active power to apparent power defines the power factor \( PF \), i.e. Equation (2.57). The term \( PF_1 \) is used to assess the power flow at fundamental conditions, while \( PF \) is used to indicate the degree of utilization of the line.
Power Theory Proposed by L. Czarnecki

The development of the Currents’ Physical Components (CPC) power theory started in [16] and is based on the decomposition of the supply current into active current, $i_a$, reactive current, $i_r$, and scattered current $i_s$. The CPC power theory is developed in the frequency domain and the author’s idea is to decompose the non-sinusoidal currents into components that can be related to a physical phenomenon. However, the author recognizes these are mathematical entities and although they do not exist in reality, they can be related to a physical phenomenon. For instance, the active current can be related to the current on a resistor, the reactive current can be related to the $90^\circ$ shift from the voltage and the scattered current to the scattered value of the conductance with harmonics. The theory actually splits the current into more components but these components are not required in this dissertation. Equations (2.59) to (2.61) provide the definition of each current component. Since the active current is the current of an equivalent resistive load that has the same active power at the same voltage, then it becomes necessary to find the value of this equivalent resistive load. Equation (2.58) specifies the means for calculating this quantity and is identical to the one in Fryze’s model. Each current component is responsible in turn for a power component. Therefore the active current, $i_a$, factors in the calculation of the active power $P$, the reactive current, $i_r$, for the reactive power $Q_r$, and the scattered current $i_s$, for the scattered power $D_s$. Expressions from (2.62) to (2.65) provide the means of calculating each power component. Finally, Equation (2.66) states that the geometric addition of $P, Q_r$ and $D_s$ yields the apparent power. It can be seen that the CPC power theory is heavily influenced by Fryze’s model; this might be explained due to the fact that Czarnecki was a former student of Fryze. Nevertheless, Czarnecki’s model introduces two new quantities,

\[ PF_1 = \frac{P_1}{S_1} \]  \hspace{1cm} (2.56)

\[ PF = \frac{P}{S} \]  \hspace{1cm} (2.57)
namely, the scattered current and the scattered power, which is a direct consequence of the scattered current.

\[ G_e = \frac{P}{\|V\|^2} \]  
(2.58)

\[ \|i_a\| = \frac{P}{\|V\|} \]  
(2.59)

\[ \|i_s\| = \sqrt{\sum_{n \in M^o} (G_n - G_e)^2 V_n^2} \]  
(2.60)

\[ \|i_r\| = \sqrt{\sum_{n \in M} B_n^2 V_n^2} \]  
(2.61)

\[ P \equiv \|v\| \|i_a\| \]  
(2.62)

\[ D_s \equiv \|v\| \|i_s\| \]  
(2.63)

\[ Q_r \equiv \|v\| \|i_r\| \]  
(2.64)

\[ S \equiv \|v\| \|i\| \]  
(2.65)

\[ S^2 = P^2 + D_s^2 + Q_r^2 \]  
(2.66)

2.6 Other Power Theories

It will be misleading to leave the impression that the above four power theories cover the numerous approaches scholars and scientists have taken to solve this unresolved, century-old problem of constructing a power theory suitable for non-sinusoidal conditions. However, mentioning all the work accumulated over more than a century attempting to solve this problem might be the subject of a large textbook. Thus, a short
list of the most influential authors to this dissertation is provided below; it is recognized however, that this approach is unfair to the contributions of other authors and it is hoped a venial judgment from these authors might be extended. In an effort to name, within the given constraint, the largest possible number of contributing authors, no description is provided for their power theories but only the name of their power theory or a phrase that highlights their contribution. Baggini [2] provides a more descriptive list.

1. A. Nabae and T. Tanaka – Powers based on instantaneous space vector
2. W. Shepherd and P. Zakikhani – Definition of reactive power
3. N. L. Kuster’s and M J M Moore – Inductive and capacitive current
5. D. Sharon – Reactive power definitions
6. M. D. Slonim and J D Van Wyck – Definition of active, reactive and apparent powers
7. A. E. Emanuel – Definitions of apparent power
8. F. Z. Peng and J S Lai – Generalized instantaneous reactive power theory
9. Ferrero, Superti-Furga – The Park power theory
10. Rossetto and Tenti – Instantaneous orthogonal currents
11. Peng – generalized non-active power theory
12. Willems – Instantaneous voltage and current vectors
13. P. S. Fillipski– Elucidation of apparent power and power factor
14. E. H. Watanabe – Generalised theory of instantaneous powers α-β-0 transformation
15. N. LaWhite, M. D. Ilić – Vector space decomposition of reactive power
16. F. Ghassemi – Definition of apparent power based on modified voltage
18. Zhang– Universal instantaneous power theory
20. M. T. Haque – Single phases p-q theory
21. A. Menti, T. Zacharias, and J. Milias-Argitis – Introduced the framework of Geometric Algebra to non-sinusoidal power theory
23. D. Xianzhong, L Guohai and G Ralf – Generalised theory of instantaneous reactive power for multiphase system
24. Shin-Kuan Chen and G W Chang – Instantaneous power theory based on active filter
25. L. M. Dalgerti – Concepts based on instantaneous complex power approach

2.7 Chapter Summary

A succinct overview of the classical sinusoidal power theory is provided at the beginning of the chapter and a number of questions without answers are left as a preamble to Chapter Three at the end of section 2.1. A concise description of Budeanu’s power theory, Fryze’s power theory, the power theory proposed by the IEEE standard 1459-2010 and the CPC power theory are provided in sections 2.2, 2.3, 2.4 and 2.5. The final section mentions the contributions of a number scholars as a recognition to their work as some of them have dedicated a good portion of their scientific life to resolve this unresolved, century-old challenge.
Chapter Three: Understanding the Root Cause of the Problem for Non-sinusoidal Power Theories

3.1 The Limitation of Steinmetz’s Model in Non-sinusoidal Circuit Analysis

Prior to 1893 the analysis of electric circuit involving alternating voltages and currents was a daunting task; however, the work of Steinmetz [17] and Kelleny [18] brought an end to that challenge. For more than a century, Steinmetz’s circuit analysis technique has shown time and again to be invaluable. However, a simple analysis of either one of the circuits in Figure 3.1 reveals limitations of the Steinmetz’s technique when applied to circuits with non-sinusoidal voltages and currents. Regrettably, these limitations have been overlooked. Although Steinmetz’s technique allows determining the appropriate values of voltages and currents in non-sinusoidal conditions; the technique requires a separate analysis for each harmonic and each analysis yields a set of results; unfortunately, in the frequency-domain, these different sets of results are mathematically unrelated. The analysis of circuits a and b of Figure 3.1 elucidates this point; hereinafter circuits a and b are referred to as circuits 3.1a and 3.1b. In both circuits, the voltage source is given by Equation (3.1), where \( \omega = 1 \text{ rad/sec} \). Although the analysis of one circuit is sufficient to illustrate the limitations of Steinmetz’s technique, the analysis of the two circuits is helpful to illustrate the consequences on the traditional definition of apparent power.

\[
v(t) = 100\sqrt{2} \{\sin(\omega t) + \sin(3\omega t)\} \tag{3.1}
\]
Figure 3.1   Circuits used to identify the limitation of Steinmetz’s technique in non-sinusoidal conditions

Steinmetz’s technique implies the existence of two domains and a transformation operation linking these two domains. Thus, the sinusoidal voltages and current signals described in the time-domain, need to be transformed to the frequency-domain via a transformation operation. Equation (3.3) defines the phasor transformation $\mathcal{P}$ for a sinusoidal signal given by Equation (3.2) [12]. Thus,

$$v(t) = \sqrt{2}V \cos(\omega t + \alpha)$$  \hspace{1cm} (3.2)

$$\mathcal{P}\{v(t)\} = Ve^{j\alpha} = \hat{V}$$  \hspace{1cm} (3.3)

In Equation (3.3) $V$ is the RMS value of $v(t)$, $\hat{V}$ represents the phasor transform of the sinusoidal signal $v(t)$ and $\alpha$ is the phase angle measured always with respect to the positive real axis of a Complex plane. The inverse phasor transformation of the phasor $\hat{V}$ is,

$$\mathcal{P}^{-1}\hat{V} = \Re\{\sqrt{2}Ve^{j\omega t}\} = \Re\{\sqrt{2}Ve^{j(\omega t + \alpha)}\} = v(t)$$  \hspace{1cm} (3.4)

A phasor quantity can be interpreted as a snapshot of a rotating complex exponential taken at time $t = 0$ [19]. Phasors obey the laws of operation of complex numbers; that is, phasors conform to the algebra of complex numbers.
The first limitation of the Steinmetz’s technique appears when attempting to transform Equation (3.1) to the frequency-domain using the phasor transformation $\mathcal{P}$. Notice that the signal $v(t)$ cannot be transformed entirely to the frequency-domain; only parts of the signal can be transformed. This limitation can be seen as inconsequential because it is always possible to appeal to the principle of superposition. This principle states that: for all linear systems, the net response caused by two or more stimulus is the sum of the responses which would have been caused by each stimulus individually [20]. Unfortunately, when only parts of the signal $v(t)$ are transformed to the frequency-domain an ambiguity then results as Equation (3.5) below shows.

$$\begin{align*}
\frac{100\sqrt{2} \sin(\omega t)}{100\sqrt{2} \sin(3\omega t)} & \rightarrow \mathcal{P} \rightarrow 100e^{j\frac{\pi}{2}} = -j100
\end{align*}$$

The ambiguity results because the same complex number represents two different phenomena. Note that both, $\mathcal{P}\{100\sqrt{2} \sin(\omega t)\} = \tilde{V}_1$ and $\mathcal{P}\{100\sqrt{2} \sin(3\omega t)\} = \tilde{V}_2$ can be considered to be a snapshots of two complex exponentials rotating at different frequency; thus, while the complex number $\tilde{V}_1 = -j100$ is a snapshot of an entity rotating at frequency $\omega$, the same complex number $\tilde{V}_2 = -j100$ is also a snapshot of another entity that rotates at frequency $3\omega$. Thus, $\tilde{V}_1 = \tilde{V}_2$ and at the same time $\tilde{V}_1 \neq \tilde{V}_2$ as $\tilde{V}_1$ and $\tilde{V}_2$ are snapshots of quantities that rotate at different frequencies; consequently, $\tilde{V}_1 + \tilde{V}_2$ cannot be performed, although the algebra of complex numbers allows this addition. This ambiguity impedes defining the operation of addition in the frequency-domain for time-domain signal of different frequencies. Consequently, this limitation renders the principle of superposition inapplicable in the frequency-domain.

As a result of the inapplicability of the superposition principle in the frequency-domain, an expression for the current or the voltage in a circuit branch cannot be attained. This second anomaly is traditionally considered to be inconsequential because the magnitude of the voltage and the current in a circuit branch or element can be found by
transferring back all the partial results to the time-domain. Equation (3.6) provides the magnitude of a time-domain signal composed by a composite of sinusoidal signal of different frequency. Thus,

\[
\begin{align*}
\text{if} & \quad x(t) = \sum_{h=1}^{n} A_h \sqrt{2} \sin(\omega_ht + \alpha) \\
\text{then} & \quad \|x(t)\| = \frac{1}{\sqrt{T_0}} \int_{-T_0/2}^{+T_0/2} x^2(t) \, dt = \sqrt{\sum_{h=1}^{n} A_h^2}
\end{align*}
\] (3.6a) (3.6b)

In Equation (3.6) \( x(t) \) can represent a voltage signal \( v(t) \) or a current signal \( i(t) \) and \( T_0 \) denotes the signal’s period. Since the magnitude of a time-domain signal such as \( v(t) = \sum_{h=1}^{n} V_h \sqrt{2} \sin(\omega_ht + \alpha) \) involves the geometric addition of frequency-domain quantities, i.e. \( V_1^2 + V_2^2 + \cdots + V_n^2 \), then the scientific community has come to accept that Equation (3.7) below, belong to the frequency-domain. Thus,

\[
\|V\| = \sqrt{\sum_{h=1}^{n} V_h^2}
\] (3.7)

Considering Equation (3.7) an equation of the frequency-domain is a mathematical error for two reasons. Firstly, Equation (3.7) is a result attained in the time-domain and as such belongs exclusively to the time-domain. Secondly, as demonstrated above, the operation of addition among quantities representing time-domain signal of different frequencies, i.e. \( V_1^2 + V_2^2 + \cdots + V_n^2 \), is not defined in the frequency-domain. Thus, it is not possible to define an operation in the frequency-domain that leads to Equation (3.7). The same hold true for the current, that is, the expression \( \|I\| = \sqrt{\sum_{h=1}^{n} I_h^2} \) is a time-domain equation not a frequency-domain equation as it is currently accepted by the scientific community.

Equations (3.8) and (3.9) provide the results of the circuit analysis of circuit 3.1a. Equation (3.8) provides the results at fundamental frequency \( \omega_1 = 1 \text{ rad/sec} \), and these results are grouped into subset \( A \) while Equation (3.9) provides the results at three-fold fundamental frequency \( \omega_3 = 3\omega_1 \) and these results are grouped into subset \( B \). The
solution set is given by the union of subsets $A$ and $B$. Notice that the quantities inside each subset are signed quantities; thus, Kirchhoff’s circuits’ laws and the principle of conservation of energy, hereinafter referred to as the PoCoE, are both verifiable inside subsets $A$ and $B$. However, circuit 3.1a does not function as the union of two different circuits; on the contrary, it functions as an indivisible unit. Consequently, the PoCoE and Kirchhoff’s circuits’ laws should be verifiable for the circuit as a whole; regrettably, this is not possible with Steinmetz’s technique.

\[
\begin{align*}
\text{subset } A, \omega = \omega_1 & : \begin{cases} 
V_1^s = -j100 & S_1^s = -j5kVA \\
Y_1 = -j0.5 & Q_1^s = -j5kVAR \\
I_1^s = -50 & P_1^s = 0 \\
Q_1^c = -625VAR & Q_1^c = 5625VAR 
\end{cases} \\
\text{subset } B, \omega = 3\omega_1 & : \begin{cases} 
V_3^s = -j100 & S_3^s = 0kVA \\
Y_3 = 0 & Q_3^s = 0 \\
I_3^s = 0 & P_3^s = 0 \\
Q_3^c = -1875VAR & Q_3^c = 1875VAR 
\end{cases}
\end{align*}
\]

\[v(t)=100\sqrt{2}\sin(\omega t)\leftrightarrow V_1^s=-j100=V_3^s\leftrightarrow V(t)=100\sqrt{2}\sin(3\omega t)\]

**Figure 3.2** Circuit requiring only fundamental frequency current; (a) currents’ flow at fundamental frequency, (b) currents’ flow at three-fold fundamental frequency, (c) a unified diagram of the flow of currents.

Notice for example in Figure 3.2(c) that Steinmetz’s technique does not allow the addition of 56.25A and 18.75A to find the current through the inductor; instead, the
technique allows finding only the magnitude of the current through the inductor, i.e. \[ \|I_L\| = \sqrt{\sum_{n=1}^{n} I_n^2} = 59.29A \]. The same is true for the current through the capacitor where \[ \|I_C\| = \sqrt{\sum_{n=1}^{n} I_n^2} = 19.76A \] and the current through the voltage source where \[ \|I_S\| = \sqrt{\sum_{n=1}^{n} I_n^2} = 50A \]. Notice that due to the fact that we have magnitudes instead of signed quantities, it is impossible to corroborate Kirchhoff’s current law at node \( n \) on Figure 3.2(c), i.e. \( \|I_S\| + \|I_C\| + \|I_L\| \neq 0 \). Therefore, although Kirchhoff’s circuit laws (KCL) are valid at each instant of time even for linear and non-linear time-invariant circuits with non-sinusoidal input; it is impossible to corroborate KCL in non-sinusoidal circuits using Steinmetz’s technique. Thus, Steinmetz’s technique allows the application of KCL in the frequency-domain only at discrete frequency values, i.e. at \( \omega, 2\omega, 3\omega \ldots n\omega \).

The principle of conservation of energy states that \textit{the instantaneous rate of instantaneous volt-amperes at the input terminal is equal to the sum of the instantaneous volt-amperes at each load component [10]}]. This implies that the total volt-amperes at the load, denoted by \( S_1^a \), should be identical to the total volt-amperes at the source, denoted by \( S_2^a \). Notice that the apparent power is the only defined power quantity that can be used to estimate the total volt-amperes. As the physical meaning of the apparent power \( S \) is at least as source of controversy as pointed out in Chapter Two, perhaps a different symbol from \( S \) such as \( \forall \) should be used to denote the total volt-amperes. Nevertheless, throughout the thesis the symbol \( S \) will remain in use to denote the total volt-amperes.

According to ohm’s law, the fundamental frequency voltage denoted by \( V_1^s \) in Figure 3.2 is \( V_1^s = Z_1^{ca} I_1^{ca} \), where \( Z_1^{ca} \) is the capacitor impedance at fundamental frequency and \( I_1^{ca} \) the current through the capacitor at fundamental frequency. Alternatively, \( V_1^s \) can be attained also as \( V_1^s = Z_1^{a1} I_1^{La} \), where \( Z_1^{La} \) is the inductor impedance at fundamental frequency and \( I_1^{La} \) is the current through the inductor at fundamental frequency. Similarly, the third harmonic voltage denoted by \( V_3^s \) can be obtained as \( V_3^s = Z_3^{ca} I_3^l \) or by \( V_3^s = Z_3^{a1} I_3^l \), where \( Z_3^{ca} \) and \( Z_3^{La} \) are the capacitor impedance and the inductor impedance at three fold fundamental frequency respectively and \( I_3^l \) is the third harmonic
loop-current inside the $LC$-branch. Thus,

\begin{align*}
V_1^s &= Z_1^{ca} I_1^{ca} = Z_1^{La} I_1^{La} \\
V_3^s &= Z_3^{ca} I_3^{ca} = Z_3^{La} I_3^{La}.
\end{align*}

Notice that both circuit elements, capacitor and inductor, are supplied by the same voltage source and the total voltage across these elements must be described by the addition of the fundamental frequency voltage $V_1^s$ and the third harmonic voltage $V_3^s$ just as it is performed in the time-domain. Thus, the total voltage should be defined by $V_t^a = [V_1^s + V_3^s]$. Similarly, the total volt-amperes at the load $S_t^a$ should be calculated as the total volt-amperes at the capacitor $S_c^a$ plus the total volt-amperes at the inductor $S_l^a$. Thus,

\[ S_t^a = S_c^a + S_l^a = S_3^a = [V_1^s + V_3^s]I_1^s = 5\sqrt{2}kVA \]  

where,

\begin{align*}
S_c^a &= [Z_1^{ca} I_1^{ca} + Z_3^{ca} I_3^{ca}][I_1^{La} + I_3^{La}] \\
S_l^a &= [Z_1^{La} I_1^{La} + Z_3^{La} I_3^{La}][I_1^{La} + I_3^{La}].
\end{align*}

Equation (3.11) shows the necessity of interpreting $V_1^s$ and $V_3^s$ as two different entities to perform the operation $V_1^s + V_3^s$. This requires reformulating the transformation operation $\mathbf{P}$ given by Equation (3.3); however, this yields a framework change, a paradigm-shift in non-sinusoidal circuit analysis.

Equation (3.12) shows the necessity for laws of operation in the frequency-domain. Notice that the terms $Z_1^{ca} I_1^{ca2}$, $Z_3^{ca} I_3^{ca2}$, $Z_1^{La} I_1^{La2}$ and $Z_3^{La} I_3^{La2}$ can be computed in the frequency-domain but the products $Z_1^{ca} I_1^{ca} I_3^{ca}$, $Z_3^{ca} I_3^{ca} I_1^{ca}$, $Z_1^{La} I_1^{La} I_3^{La}$ and $Z_3^{La} I_3^{La} I_1^{La}$ cannot as there are no laws of operations for $I_1^{ca}$ and $I_3^{ca}$, $I_1^{La}$ and $I_3^{La}$ nor $V_3^s$ and $I_1^s$. Now, since no operation can be performed among the quantities of Equations (3.8) and (3.9); it is not possible to determine, for example, how to operate the quantities $V_3^s = -j100V$ and $I_1^s = \ldots$
50A in Equation (3.11). More importantly yet, without well-defined rules of operations it is impossible to know what quantity or quantities result from the operation $V_3^s I_1^s$; it is also impossible to know the result of the operation $V_3^s I_1^s$ and the difference between the operations $V_1^s I_1^s$ and $V_3^s I_3^s$ and whether $V_1^s I_1^s$ can be added to $V_3^s I_3^s$. A clear set of laws of operation for the set formed by subsets $A$ and $B$ must be establish before any analysis is carried on. However, this shows again the necessity of a paradigm-shift in non-sinusoidal circuit analysis. Now, although Equation (3.12) cannot be computed in the frequency-domain; it is a valid expression because is a direct result of the PoCoE. Equation (3.13) can be written from the last expression in Equation (3.12). Thus,

\[ V_1^s I_1^s + V_3^s I_3^s = S_3^a = 5\sqrt{2}kVA \quad \text{and} \quad V_1^s I_1^s = 5kVA \quad (3.13) \]

Notice that the quantities $V_3^s$ and $I_1^s$ are in quadrature as the box in Figure 3.2c shows. The physical interpretation from Equation (3.13) is that there are two different forms of power; the well-known reactive power denoted by $Q$ and an additional, unidentified power that will be denoted as $UP$ initially. Thus, the total volt-amperes at the load of circuit 3.1a can be written as $S_1^a = Q_1^a + UP_1^a = S_3^a$. Let’s assume for now that the traditional definition of apparent power gives the total volt-amperes. This implies that the load requires $5\sqrt{2}kVA$ and the source must supply this amount of total volt-amperes. Equation (3.11) shows that the PoCoE cannot be verified using magnitudes; verification of Equation (3.11) requires signed quantities. In summary: it is impossible to corroborate KCL and the PoCoE in non-sinusoidal circuits using Steinmetz’s technique. Without the possibility of corroborating these two significant laws of physics, it is impossible to construct a convincing power theory; this is perhaps the reason why until today, none of the existing power theories have received general approval from the scientific community.
3.2 Is the Present Definition of Apparent Power an Appropriate Quantity for Describing the Physical Phenomena of Power Flow in Non-sinusoidal Conditions?

Equations (3.14) and (3.15) provide the results of the circuit analysis of circuit 3.1b and Figure 3.3 shows the flow of currents for each harmonic. Thus,

\begin{align*}
\text{subset } A, \omega = \omega_1 \quad & \begin{cases} 
V_1^s = -j100 \\
Y_1 = 0 \\
I_1^s = 0 \\
Q_1^c = -1875\text{VAR} \\
Q_1^t = 1875\text{VAR}
\end{cases} \\
\text{subset } B, \omega = 3\omega_1 \quad & \begin{cases} 
V_3^s = -j100 \\
Y_3 = j2 \\
I_3^s = -50 \\
Q_3^c = -5625\text{VAR} \\
Q_3^t = 625\text{VAR}
\end{cases}
\end{align*}

\begin{align*}
S_1^s &= 0\text{kVA} \\
Q_1^c &= 0 \\
P_1^s &= 0 \\
S_3^s &= j5\text{kVA} \\
Q_3^c &= j5\text{kVA} \\
P_3^s &= 0
\end{align*}

\begin{align*}
(3.14a) & \quad (3.14b) \\
(3.14c) & \quad (3.14d) \\
(3.15a) & \quad (3.15b) \\
(3.15c) & \quad (3.15d)
\end{align*}

Applying the principle of conservation of energy, PoCoE, to circuit 3.1b yields,
Following the same reasoning as for circuit 3.1a, Equation (3.16c) yields,

\[
\begin{align*}
S_C^b &= [Z_1^{cb} I_1^b + Z_3^{cb} I_3^b] [I_1^c + I_3^c] \\
S_L^b &= [Z_1^{lb} I_1^b + Z_3^{lb} I_3^b] [I_1^l + I_3^l] \\
S_t^b &= S_C^b + S_L^b = S_s^b = [V_1^s + V_3^s] I_3^s = 5\sqrt{2} kVA
\end{align*}
\] (3.16a)

Note the quantities \(V_1^s\) and \(I_3^s\) are in quadrature as the box in Figure 3.3c shows. Again, the physical interpretation of Equation (3.17) is that there are two different forms of power; the well-known reactive power denoted by \(Q\) and an additional, unidentified power that will be denoted as \(UP\) initially. Thus, the total volt-amperes at the load of circuit 3.1b can be written as \(S_t^b = Q_t^b + UP_t^b = S_s^b\). Again, this implies that the load requires \(5\sqrt{2} kVA\) and the source must supply this amount of total volt-amperes.

Now, in sinusoidal situations, if the current and the voltage do not change in a branch composed by passive elements then the total volt-amperes consumed by that particular branch must remain unchanged. On the other hand, if a voltage source is loaded with two different branches such that one branch does not supply part of the current required by the other branch, then the voltage source must provide each passive branch independently with its current requirements. More importantly, if one branch consumes \(S_1 = P_1 + jQ_1\) and the second branch consumes \(S_2 = P_2 + jQ_2\) then, according to the principle of conservation of energy, when the two passive branches are tied together the total volt-amperes consumption must be \(S_t = S_1 + S_2 = (P_1 + P_2) + j(Q_1 + Q_2)\). Unfortunately, this is not always the case in non-sinusoidal situations as the circuit in Figure 3.4 shows.
Figure 3.4  Circuit 3.4 is used to demonstrate that the traditional definition of apparent power is unsuitable for estimating the total volt-amperes and for describing the physical phenomena in non-sinusoidal condition.

Since the voltage and the current are identical on each LC-branch in Figure 3.4, 3.2 and 3.3, then the total volt-amperes on each branch remains unchanged and equal to $5\sqrt{2}kVA$. On the other hand, since the current requirements on each one of the two branches in Figure 3.4 are completely different, then, neither branch can supply the current required by the other branch. Consequently, the supply source must feed independently each branch with its current requirement. Using the principle of conservation of energy leads to the conclusion that the total volt-amperes required by the load in Figure 3.4 must be calculated as the arithmetic addition of the total volt-amperes required by each LC-branch; i.e., $S_L = (5\sqrt{2} + 5\sqrt{2}) kVA = 10\sqrt{2}kVA$. However, the application of the traditional definition of apparent power yields $S_s = ||V_s||||I_s|| = 10kVA$; which is counterintuitive as this figure implies an energy loss without any physical explanation. Notice that the figure 10kVA does not explain the accounts of powers, nor does it explain how and why some power quantities eliminated each other, i.e. how and why there was elimination among $Q_i^a$, $UP_i^a$, $Q_i^b$ and $UP_i^b$.

Applying the principle of conservation of energy to the circuit of Figure 3.4 leads the following expression,
\[ S_t = S_L^a + S_L^b + S_L^b = S_s = S_t \]  \hspace{1cm} (3.18a)
\[ S_s = [V_1^s + V_3^s][I_1^s + I_3^s] \]  \hspace{1cm} (3.18b)

Again, Equation (3.18) reveals that the principle of conservation of energy cannot be verified using magnitudes; verification of Equation (3.18) requires signed quantities. It is worth noting that the network in Figure 3.4 can be seen as a MIMO system with two inputs, \( V_1^s \) and \( V_3^s \), and two outputs \( I_1^s \) and \( I_3^s \). But more importantly, it turns out that this system is decoupled [21]. Notice that while the input \( V_1^s \) affects only the output current \( I_1^s \), the input \( V_3^s \) affects only the output current \( I_3^s \). A geometric interpretation of the traditional definition of apparent power for this example provides sufficient insight to align ourselves with Filipsky’s Remark [22].

**Figure 3.5**  Possible results for the total volt-amperes of circuit 3.4 based on the two possible additions of the apparent power of its two LC-branches

The power triangle at the origin describes the power terms of the left \( LC \)-branch in Figure 3.4 while the hatched and the shaded power triangles describe the power terms of the right \( LC \)-branch. Note that the geometric addition, described by the sum of the triangle at the origin and one of the shaded triangles, introduces a subtraction among common power terms on either one of the power axis. Figure 3.5 explains why the traditional definition gives the result \( S_t = 10kVA \) instead of \( S_t = 10\sqrt{2}kVA \). In each one of the two
cases the result is in conflict with the principle of conservation of energy which, reflecting the physical phenomena, points to the arithmetic addition as the appropriate operation.

For more than a century the efforts of constructing a theory that explains the phenomenon of powers in non-sinusoidal conditions have focused on the decomposition of the apparent power. Additionally, regardless of how this quantity is decomposed it must fulfill two conditions,

\[
S = ||V|| ||I|| \\
S^2 = P^2 + Q^2 + \sum_{i=1}^{n}(\text{power term}_i^2)
\]

(3.19a)  (3.19b)

However, note that these two restrictions are directly borrowed from the sinusoidal case. It has never been proven through an energy analysis that \( ||V|| ||I|| = \sqrt{P^2 + Q^2 + \sum_{i=1}^{n}(\text{power term}_i^2)} \); on the contrary, this condition has been forced on the definition of apparent power. In summary, the circuit in Figure 3.4 demonstrates that the traditional definition of apparent power is unsuitable for estimating the total volt-amperes, either at the load or at the source, and for describing the physical phenomena in non-sinusoidal condition.

The discussion above shows that the challenge of developing a theory capable of interpreting the power phenomena in non-sinusoidal conditions has its root on the phasor transformation \( \mathcal{P} \) given by Equation (3.3). This transformation operation embeds an ambiguity and does not codify all the information in the time-domain signal; thus, these anomalies limit the amount of energy-related information attained using Steinmetz’s technique which in turn impedes a thorough energy analysis. Consequently, a new transformation operation that codifies the entire time-domain signal’s information, namely, amplitude, phase and frequency needs to be developed. Thus, a different mathematical structure, capable of performing such a task must be brought into play. The above discussions also show that a paradigm shift in non-sinusoidal circuit analysis is required. This new paradigm must provide the means to define a power quantity that is
conservative and that is inclusive for all other possible power quantities. This new paradigm must allow also the application of the superposition principle, KCL, the balance principle of the reactive power and the principle of conservation of energy in non-sinusoidal circuits.

3.2 Mathematical Weakness of Power Definitions in Sinusoidal Conditions

Electrical engineering text books, [19] for instance, take advantage of the average power definition for developing a bridge between the frequency-domain and the time-domain. The use of this bridge yields the definition of the complex power in the frequency-domain as follows,

\[ P = VI \cos \varphi \quad (3.20) \]

Since \( \varphi \) is the angle difference between the voltage and current waveforms, Equation (3.20) can be rewritten as:

\[ P = \Re \{Ve^{j(\varphi_V - \varphi_I)}\} = \Re \{Ve^{j\varphi_V}Le^{-j\varphi_I}\} \quad (3.21) \]

Notice that in Equation (3.21) \( \varphi_V \) denotes the phase angle of the voltage signal while \( \varphi_I \) denotes the current’s phase angle. Now, since \( Ve^{j\varphi_V} \) and \( Le^{-j\varphi_I} \) are indeed complex quantities and since \( Ve^{j\varphi_V} \) can be viewed as the phasor representation of the time signal \( v(t) = \sqrt{2}V \cos(\omega t + \varphi_V) \) and \( Le^{-j\varphi_I} \) as the conjugate of the phasor representation of the time signal \( i(t) = \sqrt{2}I \cos(\omega t - \varphi_I) \), then Equation (3.21) can be interpreted as the product of the phasor voltage and the conjugate of the current phasor. Thus,

\[ P = \Re \{Ve^{j(\varphi_V - \varphi_I)}\} = \Re \{VI^*\} \quad (3.22) \]

It follows from Equation (3.22) that the complex power is then defined by
\[ S = V I^* = P + jQ \]  
\[ P = \Re e \{ V I^* \} \]  
\[ Q = \Im m \{ V I^* \} \]

Notice however that Equation (3.24) is a consequence of the choice made previously for the angle difference between the voltage and current waveforms \( \phi = \phi_V - \phi_I \). Nevertheless, since \( \cos(-\phi) = \cos(\phi) \) then the opposite subtraction is also valid for the definition of the average power. Therefore, the complex power defined also as \( S = V^* I \) is correspondingly a valid expression, but more importantly, the concept of complex power emerges more as a lucky mathematical coincidence [12].

In contrast to the above, using the fact that the product operation in the time-domain corresponds to the operation of convolution in the frequency-domain [11] demonstrates that the reactive power defined as \( Q = VI \sin \phi \) cannot be obtain in the frequency-domain, nor can it be found if the Fourier transform is applied to Equation (2.3). Both results yield the value of \( P = VI \cos \phi \). These results are not in line with the well-known fact that electrical circuit analysis must provide identical results in both domains, as well as the fact that different well defined methods should provide equal results.

Although the process to determine the complex power described by Equations (3.20) to (3.23) is widely accepted and used in almost every book devoted to electric circuits, this approach yields an ambiguity. Notice that Equation (3.20) describes unquestionably a scalar number in the time-domain and the end result i.e. Equation (3.23), yields a complex number in the frequency-domain. This result is unfortunate as it maps scalar numbers in the time-domain with complex numbers in the frequency-domain. Notice that a resistor, which is a scalar number in the time-domain, is mapped in the frequency-domain as a scalar number also; however, according to the result of Equation (3.23) a resistor should
be represented as a complex number. This unfortunate result leads to conclude that there is a fault in the process described by Equations (3.20) to (3.23) and as a consequence it cannot be accepted despite its general acceptance by the scientific community. Thus, a new theory, which takes care of this issue as well as the issues pointed by [11], must be developed. The actual flaw in the process described by Equations (3.20) to (3.23) is that the process involves an alternative transformation function besides the one defined by Equations (3.3) and (3.4). Notice that by introducing an alternative bridge (transformation function) between the two domains an erroneous result with a valid appearance appears. This result should not be a surprise as any illegal document always results from using an alternative entity (bridge) and always has a valid appearance. The appropriate equation for the complex power in the frequency-domain must result from applying the transformation function to the power expression in the time-domain, i.e., applying Equation (3.3) to Equation (2.3). However, such a task cannot be accomplished using Equation (3.3).

The circuit examples in Figures 3.1 to 3.4 and the above discussion show that:

- The traditional definition of apparent power does not always measure the total volt-amperes.
- The circuit analysis technique developed by Steinmetz has severe limitations when applied to circuits with non-sinusoidal waveforms, and
- There are a number of concepts in the frequency-domain that require a more rigorous mathematical treatment, e.g., the concept of complex power, operation among quantities for which no laws of operations have been defined.

### 3.3 Chapter Summary

This chapter exposes the current limitation of Steinmetz’s circuit analysis technique when used in the analysis of circuit with non-sinusoidal voltages and currents. An example is provided to show that the traditional definition of apparent power does not
always adequately represents the total volt-amperes. This inconsistency of the apparent power leads to conclude that this quantity is not appropriate for describing the physical phenomena of the flow of power in non-sinusoidal conditions. A number of concepts in the frequency-domain that require a more rigorous mathematical treatment are also exposed. The chapter ends emphasising the necessity of a paradigm-shift in the framework used to analyse non-sinusoidal circuit.
Chapter Four: A New Circuit Analysis Approach Using Geometric Algebra

4.1 Rudiments of Geometric Algebra

In general, an algebra could be thought of as a mathematical structure consisting of a non-empty set of elements $A$, a set of operations and a set of axioms. The name Clifford algebra, which is another name for geometric algebra, honors the English philosopher and mathematician William Kingdom Clifford who was one of the first to recognize the work of Guther Grassman, a German school teacher. Clifford unified the ideas of Grassmann and Hamilton in a work that he named geometric algebras. By introducing a new geometric product Clifford demonstrated how Hamilton’s quaternion algebra could be included into the structure developed by Grassmann. The versatility of geometric algebra resides in its ability to place tensors and spinors at the same level, an important fact in the theory of relativity.

In order to understand Clifford’s discoveries it is useful to examine first some basic definitions of vector spaces; however, it is not the intent of this section to provide an in-depth study of geometric algebra as this is out of the scope of this dissertation. References [23] to [31] provide an excellent background for introductory and in-depth study.

**Definition 1.** Let $F = \{ \alpha, \beta, \gamma, \lambda, \mu \ldots \}$ be a field. The set of $V = \{ a, b, c, d, e \ldots \}$ is a vector space or linear over $F$ if and only if

1. $V$ is an Abelian group
2. $FXV \to V$ is an external law of composition such that $V$ is a group with operators and $F$ is a dominion with distributive operators, i.e. $\mu(a + b) = \mu a + \mu b$
3. $(\lambda + \mu)a = \lambda a + \mu a$
4. $(\lambda \mu)a = \lambda(\mu a)$
5. $1a = a$ where $a$ is the multiplicative module of $F$

**Definition 2.** A set of vectors $V = \{x_1, x_2, \ldots, x_n\}$ form a basis if the following conditions are met.

1. The set $V$ spans all of their vector space
2. The set $V$ is linearly independent. The set of vectors $V = \{x_1, x_2, \ldots, x_n\}$ is linearly independent if $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$ holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

A base is a set of linearly independent vectors $\{x_j\}$ which generate the space. In particular, if the set of vectors $\{x_j\}$ satisfy $x_j x_k = C_{jk} \delta_{jk}$ and $x^\mu x_\nu = C_{\nu}^\mu \delta_{\mu}^\nu$ where $\delta_{\mu}^\nu$ is the Kronecker delta, the Einstein summation is used and $C_{jk}$ and $C_{\nu}^\mu$ are constant not necessarily equal to one, then the base is orthogonal.

**Definition 3.** A vector space, different from $\{0\}$, that has a base with $n$ elements is an $n$-dimensional space. For the spaces with finite dimension, the number of elements of the base is nothing more than the cardinal of the set. The dimension of a linear vector space is the largest possible number of linearly independent vectors that can be taken from that space.

**Definition 4.** Let $V$ and $W$ be two vector spaces over the field $F$. $V$ is isomorphic to $W$ if and only if an isomorphism $\phi$ exist between the two groups, i.e., for all $a$ and $b$ in $V$ $(a + b)\phi = a\phi + b\phi$ and for all $F$, $(\lambda a) \phi = \lambda(a\phi)$

**Definition 5.** An inner product of a space vector over $\mathbb{R}$ is a function $(\cdot, \cdot)$ of $V \times V \to \mathbb{R}$ such that

1. $a \cdot b = b \cdot a, \forall a, b \in V$
2. $(\lambda a + \mu b) \cdot a = \lambda(a \cdot c) + \mu(b \cdot c), \forall a, b \in V, \lambda, \mu \in \mathbb{R}$
3. $a \cdot a > 0, \text{ if } a \neq 0$
In $\mathbb{R}^n$ for $a = \{x_1, x_2, \ldots, x_n\}$ and $b = \{y_1, y_2, \ldots, y_n\}$ the following equality follows

$$a \cdot b = \sum_{i=1}^{n} x_i y_i$$  \hspace{1cm} (4.1)

**Definition 6.** An Euclidean space is a vector space over $\mathbb{R}$ endowed with an inner product.

**Definition 7.** In an Euclidean space the length or the norm of a vector $a$, denoted as $\|a\|$, is defined by the equality,

$$\|a\| = \sqrt{a \cdot a}$$  \hspace{1cm} (4.2)

**Definition 8.** In an Euclidean space two vectors are orthogonal or perpendicular if and only if $a \cdot b = 0$

A base is orthonormal if is orthogonal and if each vector has a unit norm. In an Euclidean space it is always possible to find a unit vector $b$ as long as $b$ is different from zero. This is described by the equality,

$$b = \frac{a}{\|a\|}$$  \hspace{1cm} (4.3)

As in Equation (4.3) $\|b\| = 1$ then an orthonormal base can be attained from an orthogonal base by dividing each element of the base by its norm. At this point the stage is set to introduce the geometric algebra or Clifford’s algebras.

Taking any orthonormal base such as $\{\sigma_1, \sigma_2\}$ in $\mathbb{R}^2$ and a vector $a = x\sigma_1 + y\sigma_2$ then, according to definition 7, the length or the norm of this vector is given by

$$\|a\| = \sqrt{x^2 + y^2}$$  \hspace{1cm} (4.4)
The product of $a$ by $a$, i.e., $a^2$ can be established as the square of the length of $a$. Symbolically this relationship can be expressed as,

$$a^2 = \|a\|^2$$  \hspace{1cm} (4.5)

However,

$$(x\sigma_1 + y\sigma_2)^2 = x^2\sigma_1^2 + xy(\sigma_1\sigma_2 + \sigma_2\sigma_1) + y^2\sigma_2^2$$  \hspace{1cm} (4.6)

Therefore, Equation (4.6) yields Equation (4.7) if and only if the conditions on Equation (4.8) are met. Thus,

$$(x\sigma_1 + y\sigma_2)^2 = x^2 + y^2$$  \hspace{1cm} (4.7)

$$\sigma_1^2 = \sigma_2^2 = 1 \quad and \quad \sigma_1\sigma_2 = -\sigma_2\sigma_1$$  \hspace{1cm} (4.8)

Applying the laws of algebra and the conditions given by Equation (4.8) yields the following result for $(\sigma_1\sigma_2)^2$,

$$
\begin{align*}
(\sigma_1\sigma_2)^2 &= (\sigma_1\sigma_2)(\sigma_1\sigma_2) \\
(\sigma_1\sigma_2)^2 &= \sigma_1(\sigma_2\sigma_1)\sigma_2 \\
(\sigma_1\sigma_2)^2 &= \sigma_1(-\sigma_1\sigma_2)\sigma_2 \\
(\sigma_1\sigma_2)^2 &= -(\sigma_1\sigma_1)^2(\sigma_2\sigma_2)^2 \\
(\sigma_1\sigma_2)^2 &= -(1)(1) \\
(\sigma_1\sigma_2)^2 &= -1
\end{align*}
$$  \hspace{1cm} (4.9)

The product $\sigma_1\sigma_2$ is an element, whose square is equal to $-1$; thus, this element is neither a vector nor a scalar and is foreign to the standard vector space. This product generates a new type of unit termed a bivector and represents an oriented plane area. As geometric algebra is a multidimensional system, the geometric algebra of the plane is a subset of geometric algebra; this two dimensional space is spanned by the base $A =$
{1, σ₁, σ₂, σ₁σ₂}, where the entity σ₁σ₂ is the highest grade element of the algebra and as such is also known as the pseudoescalar element. As a geometric object σ₁σ₂ is characterized by its dimension, its direction and its magnitude interpreted as an area since it is a two-dimensional element. The entity provides then the magnitude of an oriented plane. Therefore, as Figures 4.1 and 4.2 show, the pseudoscalar quantity σ₁σ₂ can represent any planar shape, i.e. a square, a circle, a triangle, a parallelepiped, etc. – as long as the dimension, magnitude and direction are preserved by the bi-dimensional shape – [23-31]. This element is symbolically represented in the literature by σ₁₂, σ₁σ₂, σ₁σ₂, \( I \) or \( R \). The \( I \) notation is mostly used in connection to its pseudoscalar quality and it will be evaded here to avoid confusion with the symbol \( I \) which is already used in circuit theory to represent a current phasor. The symbols σ₁σ₂ and σ₁ ∧ σ₂ will be used here.

\[ Yσ₂ \]

\[ Xσ₁ \]

\[ σ₂ \]

\[ σ₁ \]

\[ σ₁σ₂ \]

\[ σ₂σ₁ \]

Figure 4.1  The bivector unit and its different forms.

Notice that a counterclockwise rotation of ninety degrees produces a role change of the coordinate axis since \( σ₁ \) becomes \( σ₂ \) (\( σ₁ → σ₂ \)) and \( σ₂ \) becomes \(-σ₁\), (\( σ₂ → -σ₁ \)) then the product \( σ₁σ₂ = -σ₂σ₁ \) is a natural consequence.

4.1.1 The Exterior Product

The product \( ab \), where vector \( a = (λ₁σ₁ + λ₂σ₂) \) and vector \( b = (μ₁σ₁ + μ₂σ₂) \) contains the following elements,
If the scalar part of the product \( ab \) is eliminated then a new type of product labeled as exterior product can be established and its definition can be as follows.

The exterior product is denoted by the Arabic letter \( \wedge \) instead of the more commonly known symbol \( \times \) of Gibbs’ vector algebra due to the following three reasons:

1. In Gibbs’ vector algebra the exterior product is defined for vectors only, while in geometric algebra the exterior product is defined for multivectors [26]
2. The exterior product in Gibbs’ vector algebra is a special case of the exterior product in Geometric Algebra [31]
3. The symbol \( \times \) is used in Geometric Algebra to denote the commutator product – an unknown operation in Gibbs’ vector algebra – [26], [27] [31],

The exterior product is also known as the outer product and the terminology varies among authors, e.g. Sabbatta [26] uses the term exterior product while Perwass [31] uses the term outer product.

**Definition 9.** Given two vectors \( a = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \) and \( b = (\mu_1 \sigma_1 + \mu_2 \sigma_2) \) the exterior product \( a \wedge b \) is defined by the equality,

\[
ab = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2)(\mu_1 \sigma_1 + \mu_2 \sigma_2) \\
ab = \lambda_1 \mu_1 \sigma_1^2 + \lambda_1 \mu_2 \sigma_1 \sigma_2 + \lambda_2 \mu_1 \sigma_2 \sigma_1 + \lambda_2 \mu_2 \sigma_2^2 \\
ab = (\lambda_1 \mu_1 + \lambda_2 \mu_2) + (\lambda_1 \mu_2 - \lambda_2 \mu_1)\sigma_1 \sigma_2
\]  

The bivector \( a \wedge b \) represents the plane region composed of the parallelogram of sides \( a \) and \( b \), where the tail of vector \( b \) is placed over the tip of vector \( a \) and the direction of the two-dimensional element is found by following the direction of vectors \( a \) and \( b \), also given in the graph by the curved oriented path.
**Figure 4.2**  **Graphical representation of the exterior product**

Bivectors $a \wedge b$ and $b \wedge a$ have the same magnitude but opposite orientation. Therefore, the exterior product is anti-symmetric. In addition, for all vector $a$ the following holds.

$$a \wedge a = 0$$  \hspace{1cm} (4.12)

The exterior product obeys the left and right distributive rules with respect to addition

$$a \wedge (b + c) = a \wedge b + a \wedge c \hspace{1cm} (4.13a)$$
$$\hspace{0.5cm} (b + c) \wedge a = b \wedge a + c \wedge a \hspace{1cm} (4.13b)$$

### 4.1.2 The Clifford Product

The product $ab$ is given by the equality,

$$ab = (\lambda_1 \mu_1 + \lambda_2 \mu_2) + (\lambda_1 \mu_2 - \lambda_2 \mu_1)\sigma_1\sigma_2$$  \hspace{1cm} (4.14)

Therefore, Equation (4.14) embeds the inner and outer product since,

$$a \cdot b = (\lambda_1 \mu_1 + \lambda_2 \mu_2) \hspace{1cm} (4.15a)$$
$$a \wedge b = (\lambda_1 \mu_2 - \lambda_2 \mu_1)\sigma_1\sigma_2 \hspace{1cm} (4.15b)$$
Equations (4.14) and (4.15) can be used to define the Clifford product, also known as the geometric product. Thus,

**Definition 10.** Given two vectors \( a = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \) and \( b = (\mu_1 \sigma_1 + \mu_2 \sigma_2) \) the Clifford product \( ab \) is defined by the equality,

\[
ab = a \cdot b + a \wedge b = (\lambda_1 \mu_1 + \lambda_2 \mu_2) + (\lambda_1 \mu_2 - \lambda_2 \mu_1) \sigma_1 \sigma_2
\]  
(4.16)

Equation (4.16) shows that the geometric product is a combination of a scalar and a bivector. The scalar part of Equation (4.16) corresponds to the inner product of vectors \( a \) and \( b \) while the bivector part corresponds to the outer product of the vectors \( a \) and \( b \). The geometric product is associative, i.e., given three vectors \( a, b, c \)

\[
a(bc) = (ab)c = abc
\]  
(4.17)

### 4.1.3 The Geometric Algebra of the Plane, the \( \mathcal{G}_2 \) domain

The geometric algebra of the two dimensional space, i.e. \( \mathcal{G}_2 \), is spanned by the base \( A = \{1, \sigma_1, \sigma_2, \sigma_1 \sigma_2\} \). An element \( M \) of \( \mathcal{G}_2 \) is given by the following equality,

\[
M = \alpha_0 1 + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_1 \sigma_2
\]  
(4.18)

In Equation (4.18) \( M \) is a linear combination of the scalar \( \alpha_0 \), the vector \( \alpha_1 \sigma_1 + \alpha_2 \sigma_2 \), and the bivector \( \alpha_3 \sigma_1 \sigma_2 \). The geometric algebra of the plane, also known as the \( \mathcal{G}_2 \) algebra, is a 4-dimensional space over the reals with the following multiplication table.
Table 4.1 Multiplication Table for the Geometric algebra $\mathcal{G}_2$

<table>
<thead>
<tr>
<th>*</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_1\sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>1</td>
<td>$\sigma_1\sigma_2$</td>
<td>$\sigma_2$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$-\sigma_2\sigma_1$</td>
<td>1</td>
<td>$-\sigma_1$</td>
</tr>
<tr>
<td>$\sigma_1\sigma_2$</td>
<td>$-\sigma_2$</td>
<td>$\sigma_1$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

The element $M$ is called a multivector and is a linear combination of the above elements. It can be seen also from Table 4.1 that, in the 2-dimensional space the base elements that span the full algebra can be classified as:

$$1 \quad \{\sigma_1, \sigma_2\} \quad \sigma_1 \wedge \sigma_2$$

1 scalar 2 vectors 1 pseudoscalar

Since geometric algebra is a graded algebra, the elements of the algebra can be broken into terms of different grade [24]. The geometric product of two vectors which resulted in the sum of two elements of different grade, a scalar and a bivector is a good example of this concept.

Geometric Algebra described as an axiomatic system is closed under the operations of addition and the geometric product, is commutative under addition, is associative for both addition and geometric product; the geometric product obeys the left and right distributive rules in connection to addition. Geometric algebra has an identity element for the operations of addition and geometric product; the addition operation has an additive inverse. The scalars in the algebra are the real numbers and finally the geometric product between scalars and any other element of the algebra is commutative.
On the other hand, according to the definition, the geometric product of

$$\sigma_1 \sigma_2 = \sigma_1 \cdot \sigma_2 + \sigma_1 \land \sigma_2$$

(4.19)

It is always possible to operate vectors on the left or on the right of the bivector $\sigma_1 \land \sigma_2$. For example operating $\sigma_1 \land \sigma_2$ on the left of $\sigma_1$ and $\sigma_2$ produces the first two expressions on Equation (4.20) while operating on the right produces the last two expressions on Equation (4.20). Thus,

\[
\begin{align*}
(\sigma_1 \land \sigma_2)\sigma_1 &= (-\sigma_2\sigma_1)\sigma_1 = -\sigma_2 & (4.20a) \\
(\sigma_1 \land \sigma_2)\sigma_2 &= (\sigma_1\sigma_2)\sigma_2 = \sigma_1 & (4.20b) \\
\sigma_1(\sigma_1 \land \sigma_2) &= \sigma_1(\sigma_1\sigma_2) = \sigma_2 & (4.20c) \\
\sigma_2(\sigma_1 \land \sigma_2) &= \sigma_2(-\sigma_2\sigma_1) = -\sigma_1 & (4.20d)
\end{align*}
\]

It can be assumed that the coordinate system formed by $\sigma_1$, $\sigma_2$ is a dextro coordinate system; therefore, left multiplication by $\sigma_1\sigma_2$ over a vector rotates the vector $90^\circ$ in the clockwise direction. Operating $-\sigma_2\sigma_1$ on the right produces a $90^\circ$ rotation in the counterclockwise direction. The roll of $\sigma_1\sigma_2$ is also seen from Table 4.1 by computing $(\sigma_1\sigma_2)\sigma_2$ and $\sigma_1(\sigma_1\sigma_2)$ which results in $\sigma_1$ for the former and $\sigma_2$ for the latter. In both cases multiplication by $\sigma_1\sigma_2$ yields a $90^\circ$ rotation. When $\sigma_1\sigma_2$ is operated twice on the right or on the left the result is a $180^\circ$ rotation which implies a change in the sense of the vector which is equivalent to multiply by $-1$ as $(\sigma_1\sigma_2)^2 = -1$.

The similitude between imaginary unit of the complex numbers algebra and the pseudoscalar unit of the geometric algebra $G_2$ goes even further. Notice that since the relative direction of two vectors is characterized by the directed arc that relates them [24, 26], then $\sigma_1\sigma_2$ can be interpreted as representing the arc from $\sigma_1$ to $\sigma_2$, which in a Cartesian plane results in an arc of length $\pi/2$ radians. Both, the unit pseudoscalar $\sigma_1\sigma_2$ and the imaginary quantity $j$ viewed as operators perform a $90^\circ$ rotation; however, while the imaginary unit $j$ is commutative, the unit pseudoscalar $\sigma_1\sigma_2$ is not.
The geometric product as provided by definition 10 can be rewritten in a slight different form. Notice that the product $a \cdot b$ denotes the regular inner product of two vectors with magnitude calculated as $a \cdot b = \|a\|\|b\| \cos \varphi$, where $\varphi$ is the angle between vectors $a$ and $b$. On the other hand, the product $a \wedge b$ depicts the outer product of two vectors representing an oriented plane with magnitude equal to $a \wedge b = \|a\|\|b\| \sin \varphi$. Therefore the geometric product of two vectors can be written as:

$$ab = \|a\|\|b\| \cos \varphi + \|a\|\|b\| \sin \varphi (\sigma_1 \sigma_2)$$

(4.21)

In an Euclidean space, the magnitude of a vector is typically defined by the $L_2$ norm and is represented by the symbol $\| \|$ [31]; therefore, in Equation (4.21) the symbol $|$ is not used because the symbol $\| \|$ is the appropriate one. Examination of Equation (4.21) shows that the result is basically a disguised form of a complex number, where the scalar piece corresponds to the real part of the complex number and the pseudoscalar portion corresponds to the imaginary component. Also, Equation (4.21) shows the relation between the two-dimensional geometric algebra and the algebra of the complex numbers. The combination between a scalar and a bivector can be assimilated as a complex number through the equality,

$$Z = \lambda + \mu (\sigma_1 \sigma_2) = \lambda + j\mu$$

(4.22)

### 4.1.4 Spinors and the Reversion Operation

A spinor is defined as a quantity obtained by the geometric product of two vectors in the real plane [26]; spinors perform a rotation-dilation on the vector they operate on. The amount of rotation is given by the first expression in Equation (4.23), while the second expression gives the amount of dilation. Thus,

$$\theta = \tan^{-1} \left( \frac{\|a \wedge b\|}{\|a \cdot b\|} \right)$$

(4.23a)

$$ab = \sqrt{\|a\|^2 \|b\|^2}$$

(4.23b)
A spinor can then be interpreted geometrically as a directed arc on a circle of radius $\|ab\|$. By convention a positive rotation of a plane corresponds with a counterclockwise rotation [26, 28, 30]. Therefore, if the arc from vector $a$ to vector $b$ is positive then the arc from vector $b$ to vector $a$ is negative, so the geometric product of $ab$ and $ba$ is given by,

\[
ab = a \cdot b + a \wedge b = ||a||||b|| \cos \varphi + ||a||||b|| \sin \varphi (\sigma_1 \sigma_2) \\
ba = a \cdot b - a \wedge b = ||a||||b|| \cos \varphi - ||a||||b|| \sin \varphi (\sigma_1 \sigma_2)
\]  

(4.24a) (4.24b)

Therefore, reversing the order of the geometric product corresponds to the operation of complex conjugate of complex numbers, i.e. if

\[
\begin{align*}
(Z = ab, \\
\text{then} \\
Z^* = ba \\
\|Z\|^2 = ZZ^* = (ab)(ba) = a(ab)a = ||a||^2||b||^2 = ||a||^2 ||b||^2
\end{align*}
\]  

(4.24a) (4.25b)

(4.25c)

Where $Z^*$ denotes the reverse of $Z$ and is identical to complex conjugate and $\|Z\|$ denotes the modulus of $Z$. Note that when both sides of the equation $Z = ab$ are multiplied on the right side by $a^{-1}$ the result yields,

\[
Z^* a^{-1} = b a a^{-1} = b \quad \text{where} \quad a^{-1} = \frac{1}{a} = \frac{a}{a^2} = \frac{a}{\|a\|^2}
\]  

(4.26a) (4.26b)

Since $a$ is a generic vector of the algebra then Equation (4.26) implies that geometric algebra allows vector division, an operation that it is not defined in the standard vector algebra. The concept of spinors as described above allows a different form of classifying the elements of the geometric algebra $G_2$ into vectors and spinors in contrast to scalars, vectors and pseudoscalar. It also allows representing any multivector $M$ in $G_2$ as,

\[
M = a + S
\]  

(4.27)
Where the vector \( a = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 \) and the spinor \( S = \alpha_0 1 + \alpha_3 \sigma_1 \sigma_2 \); it is easy to see then that vectors are odd multivectors (elements of odd grade) while spinor are even multivectors (elements of even grade). The geometric algebra \( \mathcal{G}_2 \) can then be expressed as the sum of two linear spaces [26]:

\[
\mathcal{G}_2 = \mathcal{G}_2^o + \mathcal{G}_2^e \tag{4.28}
\]

Where \( \mathcal{G}_2^o \) denotes the two-dimensional linear vector space while \( \mathcal{G}_2^e \) denotes the two-dimensional linear space of spinors, both closed under multiplication but only \( \mathcal{G}_2 \) being a vector space [26]. The latter implies that complex numbers are a subset of the geometric algebra \( \mathcal{G}_2 \) which is a four-dimensional linear space containing a vector space but it is not a vector space.

### 4.2 A New Circuit Analysis Approach via Geometric Algebra, a Paradigm Shift

The use of geometric algebra, as a tool to explain the power phenomena in non-sinusoidal conditions, was introduced in 2007 by A. Menti, T Zacharias and J. Milias-Argitis [32]; since then, other authors have used geometric algebra to analyze, decompose and interpret established concepts of the apparent power [32-37]. However, in this dissertation, geometric algebra is used to develop first an alternative approach to analyze circuits which serves as the platform to develop a conservative, non-sinusoidal power theory [40-42].

As complex quantities can codify only the amplitude and the phase of sinusoidal signal in the time domain then a different mathematical tool capable of codifying also the frequency is necessary. Equation (4.29) below provides the transformation function from the time domain to the \( \mathcal{G}_x \) domain [41, 40].
In Equation (4.29) \( \Lambda_{i=2}^{n+1} \sigma_i \) and \( \Lambda_{i=1,i\neq2}^{n+1} \sigma_i \) denote the geometric product of \( n \) unit base vectors and \( X \) is an RMS value. Notice that multiplying any term in Equation (4.29) by the bivector \( \sigma_1\sigma_2 \) performs a 90° rotation; however, the rotation’s direction depends on the bivector’s position. Thus, \( \sigma_1\sigma_2[\sigma_2\sigma_3\sigma_4] = \sigma_1\sigma_3\sigma_4 \) but \( [\sigma_2\sigma_3\sigma_4]\sigma_1\sigma_2 = -\sigma_1\sigma_3\sigma_4 \). Note that although the operator \( j \) of the complex algebra also performs rotations, the geometric quantity \( \sigma_1\sigma_2 \) outshines the operator \( j \) in versatility and scope.

It is important stressing that the transformation operation defined by Equation (4.29) overcomes the ambiguity found in the transformation operation given by Equation (3.3). Consequently this guarantees the applicability of the principle of superposition. Notice also that in contrast to the transformation function given by Equation (3.3), where the frequency of the time-domain’s signal cannot be codified in the frequency-domain, Equation (4.29) codifies the frequency of the time-domain signal through the grade of the unit base vector.

### 4.2.1 Vector Rotations at Constant Angular Velocity

According to Jancewicz [29], the uniform rotation of a vector \( \mathbf{V} \) with constant angular velocity \( \mathbf{\hat{\omega}} \) may be expressed as the function,

\[
\nu(t) = e^{-\frac{1}{2} \mathbf{\hat{\omega}} t} \mathbf{V} e^{\frac{1}{2} \mathbf{\hat{\omega}} t}, \tag{4.30}
\]

In Equation (4.30) \( \mathbf{\hat{\omega}} \) is a 2-vector also called a bivector and is also \( \omega \) times the pseudoscalar \( \sigma_1\sigma_2 \) of the 2-dimensional geometric algebra. Although the derivative of
Equation (4.30) with respect to time can be attained following basic Calculus operations, the result is provided in [29] as \[ \frac{dv(t)}{dt} = \frac{1}{2} L (V \cdot \omega - \omega V) = V \cdot \omega L. \]

### 4.2.2 Circuit Analysis in the \( G_2 \) Domain under Sinusoidal Steady State Conditions

As the algebra of complex numbers is embedded in the two-dimensional geometric algebra, it is easier to introduce the use of geometric algebra in circuit analysis by analyzing first circuits with sinusoidal inputs. Thus, in the following subsections, only the first two entries of Equation (4.29) are required. Also, initially, the excitation signal in all the following examples is \( v(t) = \sqrt{2} A \cos(\omega t + \varphi) \) as this sinusoidal signal is sufficiently general.

#### 4.2.2.1 Circuits with Resistive Loads

The voltage-current relation in the time-domain for a purely resistive circuit is governed by the equation.

\[ v(t) = i(t)R \]  \hspace{1cm} (4.31)

Notice that in this particular case the input and the output signals are in phase; therefore, if the current is the input signal then the voltage signal is simply a scaled version of the current signal. Now, if the voltage is the input signal and is given by \( v(t) = \sqrt{2} A \cos(\omega t + \varphi) \), then it is possible to decompose \( v(t) \) as,

\[
\begin{align*}
  v(t) &= \sqrt{2} A \cos(\omega t + \varphi) \hspace{1cm} (4.32a) \\
  v(t) &= \sqrt{2} A \cos(\varphi) \cos(\omega t) - \sqrt{2} A \sin(\varphi) \sin(\omega t) \hspace{1cm} (4.32b) \\
  v(t) &= A_1 \cos(\omega t) - A_2 \sin(\omega t) \hspace{1cm} (4.32c)
\end{align*}
\]
Applying Equation (4.29) to Equation (4.32c), i.e. \( \cos(\omega t) \leftrightarrow \sigma_1 \) and \( \sin(\omega t) \leftrightarrow -\sigma_2 \), Equation (4.32c) can be written in the geometric algebra \( \mathbb{G}_2 \) domain as,

\[
V = A_1 \sigma_1 + A_2 \sigma_2 \tag{4.33}
\]

Notice that the 1-vector quantity described by Equation (4.33) can be seen as a phasor that has been decomposed along axis \( \sigma_1 \) and \( \sigma_2 \). Equation (4.33) can be expressed as a quantity that rotates at angular velocity \( \hat{\omega} \) just by applying Equation (4.30). More importantly yet, Equation (4.30) is also applicable to the current quantity \( i(t) \) of Equation (4.31). Thus, applying Equation (4.30) to each time-signal of Equation (4.31) yields,

\[
e^{-\frac{1}{2}i\hat{\omega}t} V e^{\frac{1}{2}i\hat{\omega}t} = e^{-\frac{1}{2}i\hat{\omega}t} I e^{\frac{1}{2}i\hat{\omega}t} R \tag{4.34}
\]

In Equation (4.34) \( V \) and \( I \) denote the voltage and current 1-vector quantities respectively while \( \hat{\omega} \) represents the fundamental frequency given as a multiple of the unit pseudoscalar \( \hat{\omega} = \omega \sigma_1 \sigma_2 \) and for simplicity assume \( \omega = 1 \text{ rad/sec} \). Equation (4.34) describes the rotation at constant angular velocity \( \hat{\omega} \) of the 1-vector quantities \( V \) and \( I \). A snapshot of this rotating system taken at time \( t = 0 \) yields,

\[
V = RI \tag{4.35}
\]

Equation (4.35) belongs exclusively to the \( \mathbb{G}_2 \) domain and the solution in this domain yields

\[
I = \frac{A}{R} \cos(\varphi) \sigma_1 + \frac{A}{R} \sin(\varphi) \sigma_2. \tag{4.36}
\]

The solution in the time domain can be found simply by replacing the values of \( \sigma_1 \) and \( \sigma_2 \) in time domain, i.e. \( \sigma_1 \leftrightarrow \sqrt{2}\cos(\omega t) \) \( \sigma_2 \leftrightarrow \sqrt{2}\sin(\omega t) \). As expected, voltage and current are in phase.
\[ i(t) = \frac{\sqrt{2}A}{R} \cos(\omega t + \varphi) \] (4.37)

4.2.2.2 Circuits with Pure Inductive Loads

The voltage-current relation in the time domain for a purely inductive circuit, i.e. neglecting the resistance, is governed by the equation,

\[ v(t) = L \frac{di(t)}{dt} \] (4.38)

Equation (4.38) can be described also by two quantities, (i.e. voltage \( V \) and current \( I \),) that rotate at constant angular velocity \( \omega \). Thus,

\[ e^{-\frac{1}{2} \omega t} V \ e^{\frac{1}{2} \omega t} = L \frac{d}{dt} \left( e^{-\frac{1}{2} \omega t} I \ e^{\frac{1}{2} \omega t} \right). \] (4.39)

Notice that due to Equation (4.30), Equation (4.39) is a replica of Equation (4.38). However, while Equation (4.38) offers almost no value when the system is analyzed at a particular chosen time (e.g. \( t = 0 \)), Equation (4.39) provides a framework for analyzing electric networks that is superior to the framework of phasors. Notice also that Equation (4.39) is also a time-domain equation; thus, it is also useful for transient analysis. Transient analysis is outside of the scope of this dissertation. Performing the derivation of Equation (4.39) and analyzing at \( t = 0 \) yields,

\[ V = \frac{1}{2} L (I \ \hat{\omega} - \omega I) = I \cdot \hat{\omega} L \] (4.40)

Solving for \( \hat{\omega} L \) in (4.40) yields

\[ -\hat{\omega} L = VI^{-1} \] (4.41)
Equation (4.41) provides the sinusoidal impedance $Z$ in the $G_2$ domain. Notice that in contrast to the traditional frequency-domain approach where $Z = j\omega L$; the impedance quantity $Z$ is preceded by a minus sign in the $G_2$ domain. Thus, $Z = -\hat{\omega}L = -\omega L\sigma_1\sigma_2$. Alternatively, Equation (4.41) can be written as,

$$I = (\hat{\omega}L)^{-1}V = \left(\frac{\sigma_1\sigma_2}{\omega L}\right)V$$  \hspace{1cm} (4.42)$$

Interpretation of Equation (4.42) shows that the output signal, in this case the current $I$, results from rotating and scaling the input signal $V$. The scaling of the input signal $V$ happens due to the factor $\frac{1}{\omega L}$ and as this factor is $0 < \frac{1}{\omega L} < 1$ then an actual scale down occurs. The clockwise rotation of $V$ happens because the pseudoscalar $\sigma_1\sigma_2$ is located on the left-hand side of voltage vector $V$, recall Equation (4.20). Thus, to attain the same result, but locating the pseudoscalar $\sigma_1\sigma_2$ on the right-hand side of voltage vector $V$, a minus sign needs to precede the pseudoscalar, recall Equation (4.20). This property of the pseudoscalar $\sigma_1\sigma_2$ allows rewriting Equation (4.42) as,

$$I = V(-\hat{\omega}L)^{-1} = V\left(\frac{-\sigma_1\sigma_2}{\omega L}\right)$$  \hspace{1cm} (4.43)$$

Equations (4.42) and (4.43) show that $I = YV = -YY$, where $Y = Z^{-1} = \frac{\sigma_1\sigma_2}{\omega L}$. Replacing $V$ with Equation (4.33) and preforming the rotation and scaling down bears,

$$I = -\frac{A}{\omega L} \cos(\phi)\sigma_2 - \frac{A}{\omega L} \sin(\phi)(-\sigma_1)$$  \hspace{1cm} (4.44)$$

Transferring Equation (4.44) back to the time domain yields,

$$i(t) = \frac{\sqrt{Z}A}{\omega L} \sin(\omega t + \phi)$$  \hspace{1cm} (4.45)$$
As expected from the frequency analysis, Equation (4.42) shows the current lagging the voltage by an angle of $90^\circ$.

\[
V = A_1\sigma_1 + A_2\sigma_2
\]

\[
I = \left( \frac{\sigma_1\sigma_2}{\omega L} \right) V
\]

\[
V_L = -\omega L\sigma_1\sigma_2 I
\]

**Figure 4.3** $G_2$ domain’s representation of a pure inductive circuit and its 1-vector interpretation.

Figure 4.3 describes a pure inductor circuit in the $G_2$ domain. Notice that the excitation $V$ is shifted $90^\circ$ degree in the clockwise direction and scaled down, both operations are due to the action of the bivector $\frac{\sigma_1\sigma_2}{\omega L}$.

### 4.2.2.2 Circuits with Resistor-Inductor Loads

The time domain relation of voltage and current for an RL circuit is governed by,

\[
n(t) = Ri(t) + L \frac{di(t)}{dt}
\]

(4.46)

The voltage-current relation in the $G_2$ domain can be attained either by transferring Equation (4.46) as in the previous examples, or with the aid of (4.35) and (4.40). Thus, following either approach the result is,
\[ V = RI + I \cdot \hat{\omega} L \]  

(4.47)

Solving Equation (4.47) for \( VI^{-1} \) yields

\[ VI^{-1} = R - \hat{\omega}L = R - \omega L \sigma_1 \sigma_2 = Z \]  

(4.48)

Note that the impedance \( Z \) is described by a spinor. Seeking an output/input relation, Equation (4.48) can be expressed alternatively as,

\[ I = \frac{R}{R^2 + (\omega L)^2} V + \frac{\omega L}{R^2 + (\omega L)^2} \sigma_1 \sigma_2 V \]  

(4.49)

Interpretation of Equation (4.49) shows that the output signal, in this case the current \( I \), has two components. The component in-phase with the input voltage \( V \), labeled as \( I_\parallel \), which results from scaling \( V \) and the component in-quadrature with the input voltage \( V \), labeled as \( I_\perp \) which results from rotating and scaling \( V \). Thus, \( I_\parallel = \frac{R}{R^2 + (\omega L)^2} V \)

and \( I_\perp = \frac{\omega L}{R^2 + (\omega L)^2} \sigma_1 \sigma_2 V \). The time domain solution is therefore,

\[ i(t) = \frac{\sqrt{2}AR}{R^2 + (\omega L)^2} \cos(\omega t + \varphi) + \frac{\sqrt{2}A\omega L}{R^2 + (\omega L)^2} \sin(\omega t + \varphi). \]  

(4.50)

![Figure 4.4](image)

\( G_2 \) domain’s representation of a resistive-inductive circuit and its \( 1 \)-vector interpretation.
Notice that left-hand multiplication over \( V \) by the spinor’s scalar part produces the current’s parallel component while multiplication by the spinor’s pseudoscalar part produces the orthogonal element. Figure 4.4 describes the circuit’s rotation-contraction output in the \( G_2 \) domain.

### 4.2.2.3 Circuits with Pure Capacitive Loads

The voltage/current relation in the time domain for a purely capacitive circuit is governed by the equation.

\[
i(t) = C \frac{dv(t)}{dt}
\]  

(4.51)

Equation (4.51) can be described also by two quantities, i.e. \( V \) and \( I \), that rotate at constant angular velocity \( \hat{\omega} \). Thus,

\[
e^{-\frac{1}{2} \hat{\omega} t} I e^{\frac{1}{2} \hat{\omega} t} = C \frac{d}{dt} (e^{-\frac{1}{2} \hat{\omega} t} V e^{\frac{1}{2} \hat{\omega} t})
\]  

(4.52)

Performing the derivation of (4.52) and analyzing at \( t = 0 \) yields

\[
I = V \cdot \hat{\omega} C
\]  

(4.53)

Seeking the relation \( VI^{-1} \) in (4.53) yields

\[
VI^{-1} = Z = Y^{-1} = \frac{1}{\omega C} \sigma_1 \sigma_2
\]  

(4.54)

Seeking an output/input relation, Equation (4.54) can be expressed alternatively as,

\[
I = -\omega C \sigma_1 \sigma_2 V
\]  

(4.55)
Notice that the pseudoscalar $\sigma_1\sigma_2$ located on the left-hand side of the input signal $V$ performs a rotation on the clockwise direction; however, as the quantity $\sigma_1\sigma_2$ is preceded by a minus sign this reverses the direction of the rotation. Thus, interpretation of Equation (4.55) shows that the output signal, in this case the current $I$, results from counter clock rotation and a scaling of the input signal $V$. Performing the geometric product in Equation (4.55) yields,

$$I = \omega CA \sin(\varphi)(-\sigma_1) + \omega CA \cos(\varphi)(\sigma_2). \quad (4.56)$$

The time domain solution is therefore

$$i(t) = -\sqrt{2}\omega CA \sin(\omega t + \varphi). \quad (4.57)$$

As expected from the frequency analysis, Equation (4.55) shows the current leading the voltage by an angle of 90°. Notice that (4.57) and (4.45) have opposite signs.

Figure 4.5 $\mathcal{G}_2$ domain’s representation of a capacitive circuit and its 1-vector interpretation.
4.2.2.4 Circuits with Resistor-Capacitor Loads

The time domain relation of voltage and current for an RC circuit is governed by,

\[ v(t) = Ri(t) + \frac{1}{C} \int i(t) dt + v_0 \quad (4.58) \]

In Equation (4.58) \( v_0 \) denotes the initial conditions. The voltage-current relation in the \( G_2 \) domain can be attained either by transferring Equation (4.58) as in the previous examples, or with the aid of (4.35) where \( V = RI \) and (4.55) where \( V = (\omega C)^{-1} \sigma_1 \sigma_2 I \). Thus, following either approach the result is,

\[ V = RI + (\omega C)^{-1} \sigma_1 \sigma_2 I \quad (4.59) \]

Multiplying both sides of Equation (4.59) on the right by \( I^{-1} \) yields,

\[ VI^{-1} = Z = R + \frac{1}{\omega C} \sigma_1 \sigma_2 \quad (4.60) \]

Again, as in Equation (4.48) the impedance \( Z \) of an RC circuit is described also by a spinor; however, the 2-vector components of Equations (4.48) and (4.60) have opposite directions. Thus, the series resonance condition happens when the two bivectors are summed to give a null result, i.e. \( \frac{1}{\omega C} \sigma_1 \sigma_2 - \omega L \sigma_1 \sigma_2 = 0 \), which happens when \( \omega = \sqrt{(LC)^{-1}} \). Seeking an output/input relation, Equation (4.60) can be expressed alternatively as,

\[ I = \frac{R}{R^2 + (\omega C)^{-2}} V - \frac{\omega C}{R^2 + (\omega C)^{-2}} \sigma_1 \sigma_2 V \quad (4.61) \]

Interpretation of Equation (4.61) shows that the output signal, in this case the current \( I \), has two components. The first component is the in-phase component and results simply by scaling the input signal \( V \), while the second component is the
orthogonal component and results from rotating and scaling the input signal $V$. Note that while in Equation (4.61) the rotation of the orthogonal component is in the counter clock wise direction, the rotation of the orthogonal component in Equation (4.49) is in the clock wise direction; which is a result of the minus sign in Equation (4.61). The time domain solution is therefore,

$$V = A_1 \sigma_1 + A_2 \sigma_2$$

$$I = I_\parallel + I_\perp$$

$$I_\parallel = \frac{R}{R^2 + (\omega C)^{-2}} V, \quad I_\perp = \frac{-(\omega C)^{-1}}{R^2 + (\omega C)^{-2}} \sigma_1 \sigma_2 V$$

![Diagram](image)

**Figure 4.6** $\mathcal{G}_2$ domain’s representation of a resistive-capacitive circuit and its 1-vector interpretation.

$$i(t) = \frac{\sqrt{2}AR}{R^2 + (\omega C)^{-2}} \cos(\omega t + \varphi) - \frac{\sqrt{2}A}{\omega CR^2 + (\omega C)^{-2}} \sin(\omega t + \varphi)$$  

(4.62)

Notice that left-hand multiplication over $V$ by the spinor’s scalar part produces the current’s parallel component while multiplication by the spinor’s pseudoscalar part produces the orthogonal element. Figure 4.6 describes the circuit in the $\mathcal{G}_2$ domain as well as the rotation-contraction of the input signal which yields the output signal.

### 4.2.3 Power in the $\mathcal{G}_2$ and $\mathcal{G}_N$ Domains

The mathematical weaknesses of the power’s definition in the frequency-domain for sinusoidal conditions results from using a spurious transformation function between the frequency-domain and the time domain. The power expression in a domain $D$ must result in
from the application of the transformation function between the time domain and the domain $D$, to the power expression in the time domain. In the time domain, the product of the potential variable $v(t)$ and the flow variable $i(t)$ is called power $p(t)$. Thus, if $v(t)$ and $i(t)$ are defined by Equation (2.1) and (2.2), which are repeated below for convenience, then the power in the $G_2$ domain can be found as,

$$v(t) = \sqrt{2}V \cos(\omega t + \alpha) \quad (4.63a)$$
$$i(t) = \sqrt{2}I \cos(\omega t + \alpha \pm \phi) \quad (4.63b)$$
$$p(t) = v(t)i(t) \quad (4.63c)$$
$$p(t) = \frac{VI \cos \phi [1 + \cos(2\omega t + 2\alpha)] + V I \sin \phi \sin(2\omega t + 2\alpha)}{\varphi} \quad (4.63d)$$

Performing the geometric product between the 1-vector $V$ and the 1-vector $I$ yields,

$$v(t) = \sqrt{2}V \cos(\omega t + \alpha) \rightarrow V = V_1 \sigma_1 + V_2 \sigma_2 \quad (4.64a)$$
$$i(t) = \sqrt{2}I \cos(\omega t + \alpha \pm \phi) \rightarrow I = I_1 \sigma_1 + I_2 \sigma_2 \quad (4.64b)$$
$$p(t) = v(t)i(t) \rightarrow M = VI = V \cdot I + V \wedge I \quad (4.64c)$$

Interpretation of Equation (4.65) with the aid of Equation (4.15) shows that,

$$M = (V_1 I_1 + V_2 I_2) + (V_1 I_2 - V_2 I_1) \sigma_1 \sigma_2 \quad (4.65)$$

Note the close relation between the imaginary quantity $j$ and the 2-vector quantity $\sigma_1 \sigma_2$. This relation should not be a surprise as the algebra of complex numbers is a subset of geometric algebra. However, the 2-vector quantity $\sigma_1 \sigma_2$ is more versatile as it can: act as a rotation operator not only in a two-dimensional system but also in a $N$-dimensional system and it can act also as a standalone element that can be operated – i.e. added, subtracted, multiplied, divided, etc., – with any other element such as scalars, 1-vectors, 2-vectors and any other element of an $N$-dimensional system.
Since the mathematical process described by Equation (4.64) is more rigorous than the traditional process used to attain the definition of complex power in the frequency-domain, the result attained in Equation (4.65) provides a better mathematical formality for the definition of power quantities because:

1. Equation (4.65) shows that, contrary to the frequency-domain – where the definition of Q is not a result of mathematical operation – the definition of the active average power P and the reactive power Q are both a direct result of a mathematical operation.

2. Equation (4.65) shows that, in contrast to the time domain – where a power term of \( p(t) \) is eliminated – P and Q can be defined without eliminating a power term from the power equation \( p(t) \).

3. Contrary to the frequency-domain, where the addition of P and Q is not supported, Equation (4.65) is embedded in a mathematical framework where the addition of P and Q is supported, and

4. Contrary to the definition of complex power in the frequency-domain, where \( S = VI^* = V \cdot I \) is not rigorously in harmony with the power definition of nonelectrical systems, Equation (4.65) is a definition of power that harmonize with the power definition of hydraulic, mechanical, and pneumatic systems.

Using Kuhn’s terminology [43], point four above shows external consistency. This consistency evidences how well the theory of this dissertation harmonizes with other theories dealing with the concept of power, where power is defined as the product of the potential variable (also called the across variable) and the flow variable (also called the through variable), [19], [44]. The letter \( M \) in Equation (4.65) is chosen because the power is represented by a multivector. In sinusoidal networks it contains any combination of a scalar, a vector, and a bivector; however, in non-sinusoidal systems it can involve also 3-vectors, 4-vectors and n-vectors. The letter \( P \) is kept to represent active power because its use is already recognized by the electrical engineering scientific community. The letter \( Q \)
is not used for representing the reactive power but instead the letters $CN$ as the actual reactive power is described by a Clifford Number, which includes the reactive power $Q$ represented by the bivector $\sigma_1 \sigma_2$; only in sinusoidal situations $CN = jQ$. Thus, in sinusoidal and non-sinusoidal situations, the power equation in the $G_N$ domain is given by,

\[
\mathbf{M} = \mathbf{VI} = P + CN \\
\|\mathbf{M}\| = \sqrt{\langle \mathbf{M}^2 \rangle_0}
\]  
(4.67a) \hspace{1cm} (4.67b)

In Equation (4.67b) $\|\mathbf{M}\|$ defines the norm in the $G_N$ domain, where $\mathbf{M}$ is the reverse of $\mathbf{M}$ and $\langle \cdot \rangle_0$ denotes the scalar part of $\mathbf{M}^2$. There is only one difference when calculating the value of $P$ in non-sinusoidal situations. As unit k-vectors do not always square to 1, when assessing the consumption or production of active power in an element, all the resulting scalar numbers must be multiplied by the following correction factor,

\[
f = (-1)^{k(k-1)/2}.
\]  
(4.68)

Where $k$ is the grade of the multivector. Consequently, for scalars resulting from the square of unit bivectors as $\text{in}(\sigma_3 \sigma_1)^2$ the sign changes as $k = 2$.

In sinusoidal situations there is a one-to-one correspondence for the power equation in both domains as it is seen by computing the power in the time domain and transferring the result to the $G_2$ domain. For instance, in the time domain the power for the RL circuit is,

\[
p(t) = \frac{2A^2}{R^2 + (\omega L)^2} \{R\cos^2(\omega t) + \omega L\cos(\omega t)\sin(\omega t)\}
\]  
(4.69)

Transferring Equation (4.69) to the $G_2$ domain produces the multivector power given in Table 4.2, for an RL load. Table 4.2 shows the corresponding power multivector $M$ of each one of the circuits above. The scalar part of the multivector power represents the active power while the bivector part represents the reactive power. Generally, in the sinusoidal case, the multivector power is a spinor and corresponds always to the complex power $S$ of the frequency-domain and $\|\mathbf{M}\| = \|S\|$. 
### Table 4.2  Power Multivector for Several Circuits

<table>
<thead>
<tr>
<th>Load</th>
<th>Multivector Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$M = \frac{A^2}{R}$</td>
</tr>
<tr>
<td>L</td>
<td>$M = -\frac{A^2}{\omega L} \sigma_1 \sigma_2$</td>
</tr>
<tr>
<td>RL</td>
<td>$M = \frac{A^2 R}{R^2 + \omega^2 L^2} - \frac{A^2 \omega L}{R^2 + \omega^2 L^2} \sigma_1 \sigma_2$</td>
</tr>
<tr>
<td>C</td>
<td>$M = A^2 \omega C \sigma_1 \sigma_2$</td>
</tr>
<tr>
<td>RC</td>
<td>$M = \frac{A^2 R}{R^2 + (\omega C)^{-2}} + \frac{A^2 (\omega L)^{-1}}{R^2 + (\omega C)^{-2}} \sigma_1 \sigma_2$</td>
</tr>
</tbody>
</table>

#### 4.2.4 Circuit Involving Non-sinusoidal Sources in the $G_N$ Domain

In contrast to the circuit analysis technique developed by Steinmetz; this methodology allows a simultaneous analysis of all the harmonics [40-42]. Figure 4.7 shows the results of the circuit analysis in the $G_N$ domain of the circuit in Figure 3.4 with the voltage source given by Equation (4.70).

$$v(t) = 100\sqrt{2} \{\sin(\omega t) + \sin(3\omega t)\} \quad (4.70)$$
Figure 4.7 Results for the circuit in Figure 3.4 with the new circuit analysis approach in the $G_N$ domain.

The time domain equation of the network after reducing the circuit to one $LC$-branch is,

$$v(t) + LC \frac{d^2 v(t)}{dt^2} = L \frac{d i_s(t)}{dt}$$  \hspace{1cm} (4.71a)

where  
$$L = \left( \frac{9}{16} + \frac{3}{16} \right)^{-1} = \frac{4}{3} H$$  \hspace{1cm} (4.71b)

$$C = \left( \frac{1}{16} + \frac{3}{16} \right) = \frac{1}{4} F$$  \hspace{1cm} (4.71c)

Applying Equation (4.29) to Equation (4.70) gives the RMS values of the excitation source $v(t)$ in the $G_s$ domain as $N = n + 1 = 4$. Note that the $G_N$ domain is defined by the highest harmonic available on the excitation source. In Equation (4.72) below the sub index denotes the grade of the $k$-vector and it is used to reduce the length of the equations, thus,

$$V_s = (\langle V \rangle_1 + \langle V \rangle_3)V$$  \hspace{1cm} (4.72a)

$$\langle V \rangle_1 = -100\sigma_2$$  \hspace{1cm} (4.72b)

$$\langle V \rangle_3 = 100\sigma_1\sigma_3\sigma_4$$  \hspace{1cm} (4.72c)

$$\|V_s\| = \sqrt{\langle VV \rangle_0} = 100\sqrt{2}V$$  \hspace{1cm} (4.72d)
The magnitude of the voltage source is attained by applying Equation (4.67) to the multivector $V_s$, where $\bar{V}$ is the reverse and $\langle \rangle_o$ denotes the scalar part. The key issue is how to transition Equation (4.71) from the time domain to the $\mathbb{G}_N$ domain. The answer is actually simple; by improving the concept of rotation of complex exponential of the frequency-domain. Note that thanks to the multidimensionality of geometric algebra which is not available in the algebra of complex numbers, Equation (4.70) can be written as the addition of rotation of $k$-vectors. Thus,

$$v(t) = e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_1 e^{\frac{1}{2} \bar{\omega}t} + e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_3 e^{\frac{1}{2} \bar{\omega}t}$$

(4.73)

Notice that in Equation (4.73) the 1-vector term rotates at fundamental frequency $\bar{\omega} = \omega \sigma_1 \sigma_2$, while the 3-vectors rotate at 3 fold fundamental frequency; consequently, when the excitation source involves the $h^{th}$ harmonic $h$-vector term that rotates at $\bar{\omega} = \omega h \sigma_1 \sigma_2$ is added to Equation (4.73). Thus, Equation (4.71) can be written as,

$$\left[ e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_1 e^{\frac{1}{2} \bar{\omega}t} + e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_3 e^{\frac{1}{2} \bar{\omega}t} \right] + L C \frac{d^2}{dt^2} \left[ e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_1 e^{\frac{1}{2} \bar{\omega}t} + e^{-\frac{1}{2} \bar{\omega}t} \langle V \rangle_3 e^{\frac{1}{2} \bar{\omega}t} \right] =

L \frac{d}{dt} \left[ e^{-\frac{1}{2} \bar{\omega}t} \langle I_s \rangle_1 e^{\frac{1}{2} \bar{\omega}t} + e^{-\frac{1}{2} \bar{\omega}t} \langle I_s \rangle_3 e^{\frac{1}{2} \bar{\omega}t} \right]$$

(4.74)

The equation formed exclusively by the vector terms in Equation (4.74) covers Steinmetz’s technique but, as shown previously, the technique presented in this dissertation offers a better mathematical formality. Note that, in contrast to the frequency-domain – where the principle of superposition in not applicable – the principle of superposition is embedded in Equation (4.74) and consequently all the harmonics can be examined simultaneously. More importantly, since Equation (4.74) also belongs to the time domain; it is also useful for transient analysis [42]. However, again, this type of analysis is outside of the scope of this dissertation.
The steady state analysis is performed by releasing Equation (4.74) from its time dependency which is achieved by setting \( t = 0 \). However, the differentiations must be performed first. Thus, performing the differentiations and analyzing at \( t = 0 \) yields,

\[
\langle \mathbf{V} \rangle_1 + \langle \mathbf{V} \rangle_3 - L \omega^2 [\langle \mathbf{V} \rangle_1 + 9 \langle \mathbf{V} \rangle_3] = \left( \frac{L}{2} \right) [\langle \mathbf{i}_s \rangle_1 + 3 \langle \mathbf{i}_s \rangle_3] \omega - \left( \frac{L}{2} \right) \omega [\langle \mathbf{i} \rangle_1 + 3 \langle \mathbf{i} \rangle_3]
\]

(4.75)

Equation (4.75) is simply a snapshot of rotating \( k \)-vectors taken at time \( t = 0 \). This equation belongs exclusively to the \( g_4 \) domain, and solving it for each component yields

\[
\mathbf{i}_s = \langle \mathbf{i}_s \rangle_1 + \langle \mathbf{i}_s \rangle_3; \quad \begin{cases} 
\langle \mathbf{i}_s \rangle_1 = 100 \left( \frac{L \omega^2 - 1}{\omega L} \right) \sigma_1 \\
\langle \mathbf{i}_s \rangle_3 = 100 \left( \frac{9L \omega^2 - 1}{3 \omega L} \right) \sigma_2 \sigma_3 \sigma_4
\end{cases}
\]

(4.76)

The current through the capacitor is found by adding the first terms of \( \langle \mathbf{i}_s \rangle_1 \) and \( \langle \mathbf{i}_s \rangle_3 \) in Equation (4.76); while the current through the inductor is attained by adding the second terms of \( \langle \mathbf{i}_s \rangle_1 \) and \( \langle \mathbf{i}_s \rangle_3 \). Thus,

\[
\begin{align*}
\mathbf{i}_C &= 25 \sigma_1 + 75 \sigma_2 \sigma_3 \sigma_4 \\
\mathbf{i}_L &= -75 \sigma_1 - 25 \sigma_2 \sigma_3 \sigma_4 
\end{align*}
\]

(4.77a-4.77b)

Applying the laws of current division yields the current at each one of the four branches of the circuit in Figure 4.7. Thus, using sub index \( a \) to denote the elements on the branch formed by \( C = \frac{1}{16} \) and \( L = \frac{16}{9} \) and sub index \( b \) to denote the elements on the branch formed by \( C = \frac{3}{16} \) and \( L = \frac{16}{3} \), the result yields,
Applying Kirchhoff’s circuit laws yields the current through the voltage source and the current on each LC-branch. Thus,

\[ I_s = I_C + I_L = -50\sigma_1 + 50\sigma_2\sigma_3\sigma_4 \quad (4.79) \]

\[ I_a = I^a_C + I^a_L = -50\sigma_1 \quad (4.80) \]

\[ I_b = I^b_C + I^b_L = -50\sigma_2\sigma_3\sigma_4 \quad (4.81) \]

Equations (4.80) and (4.81) show and confirm that the current requirement on each LC-branch is entirely different; thus, neither branch can aid at supplying the current requirements of the other branch. Consequently, the supply source must fulfill the current requirements of each branch as Equation (4.79) shows.

The total volt-amperes at each circuit component is given by \( M = VI = P + CN \). This leads to multiply \( V_s \) given in Equation (4.72) to each one of the four expressions in Equation (4.78). Thus,

\[
\begin{align*}
M^a_C &= V_s I^a_C = -1250(\sigma_1\sigma_2 + \sigma_3\sigma_4) & \|M^a_C\| &= 1250\sqrt{2} \\
M^b_C &= V_s I^b_C = -3750(\sigma_1\sigma_2 + \sigma_3\sigma_4) & \|M^b_C\| &= 3750\sqrt{2} \\
M^a_L &= V_s I^a_L = -1250(\sigma_1\sigma_2 + \sigma_3\sigma_4) & \|M^a_L\| &= 1250\sqrt{2} \\
M^b_L &= V_s I^b_L = -1250(\sigma_1\sigma_2 + \sigma_3\sigma_4) & \|M^b_L\| &= 1250\sqrt{2}
\end{align*}
\]
Note that Equations (3.12) and (3.16) which resulted from the principle of conservation of energy and could not be computed in the frequency-domain due to its lack of laws of operation are easily computed in the $G_{N}$ domain as $l_{1}^{ca} = 6.25 \sigma_{1}$, $l_{1}^{la} = 56.25 \sigma_{1}$, $l_{3}^{ca} = 18.75 \sigma_{2} \sigma_{3} \sigma_{4}$, $l_{3}^{cb} = 56.25 \sigma_{2} \sigma_{3} \sigma_{4}$, $l_{3}^{lb} = 6.25 \sigma_{2} \sigma_{3} \sigma_{4}$, and $l_{1}^{l} = 18.75 \sigma_{1}$. Thus,

\[
\begin{align*}
S_{c}^{a} &= [Z_{1}^{ca} l_{1}^{ca} + Z_{3}^{ca} l_{3}^{ca}] l_{c}^{a} \\
S_{c}^{b} &= [Z_{1}^{cb} l_{1}^{cb} + Z_{3}^{cb} l_{3}^{cb}] l_{c}^{b} \\
I_{c}^{a} &= l_{1}^{ca} + l_{3}^{ca} \\
I_{c}^{b} &= l_{1}^{cb} + l_{3}^{cb} \\
Z_{1}^{ca} l_{1}^{ca} &= 625 \sigma_{1} \sigma_{2} \\
Z_{1}^{cb} l_{1}^{cb} &= -1875 \sigma_{3} \sigma_{4} \\
Z_{3}^{ca} l_{3}^{ca} &= 625 \sigma_{2} \sigma_{3} \sigma_{4} \\
Z_{3}^{cb} l_{3}^{cb} &= -1875 \sigma_{1} \sigma_{2} \\
S_{c}^{a} &= M_{c}^{a} \\
S_{c}^{b} &= M_{c}^{b} \\
S_{l}^{a} &= [Z_{1}^{la} l_{1}^{la} + Z_{3}^{la} l_{3}^{la}] l_{l}^{a} \\
S_{l}^{b} &= [Z_{1}^{lb} l_{1}^{lb} + Z_{3}^{lb} l_{3}^{lb}] l_{l}^{b} \\
I_{l}^{a} &= l_{1}^{la} + l_{3}^{la} \\
I_{l}^{b} &= l_{1}^{lb} + l_{3}^{lb} \\
Z_{1}^{la} l_{1}^{la} &= -5625 \sigma_{1} \sigma_{2} \\
Z_{1}^{lb} l_{1}^{lb} &= 1875 \sigma_{3} \sigma_{4} \\
Z_{3}^{la} l_{3}^{la} &= -5625 \sigma_{2} \sigma_{3} \sigma_{4} \\
Z_{3}^{lb} l_{3}^{lb} &= 1875 \sigma_{1} \sigma_{2} \\
S_{l}^{a} &= M_{l}^{a} \\
S_{l}^{b} &= M_{l}^{b} \\
S_{t}^{a} &= S_{c}^{a} + S_{l}^{a} = S_{s}^{a} \\
S_{t}^{b} &= S_{c}^{b} + S_{l}^{b} = S_{s}^{b} \\
S_{t} &= S_{t}^{a} + S_{t}^{b} \\
\|S_{t}\| &= 10 \sqrt{2} kVA
\end{align*}
\]

Equations (4.83) and (4.84) show that the present theory used to analyze circuits in non-sinusoidal conditions gives only half of the information required to perform an energy analysis. Moreover, the amount of information is considerably reduced as the number of harmonics increases. With the appropriate laws of operation the total volt-amperes required by each LC-branch alone and together can now be found easily. Thus, $S_{l}^{a} = S_{c}^{a} + S_{l}^{a} = S_{s}^{a}$, $S_{l}^{b} = S_{c}^{b} + S_{l}^{b} = S_{s}^{b}$, $S_{t} = S_{t}^{a} + S_{t}^{b}$ and $\|S_{t}\| = 10 \sqrt{2} kVA$

\[
\begin{align*}
M_{a} &= M_{c}^{a} + M_{l}^{a} = -5(\sigma_{1} \sigma_{2} + \sigma_{3} \sigma_{4}) kVA \\
M_{b} &= M_{c}^{b} + M_{l}^{b} = -5(\sigma_{1} \sigma_{2} + \sigma_{3} \sigma_{4}) kVA \\
M_{l} &= M_{a} + M_{b} = -10(\sigma_{1} \sigma_{2} + \sigma_{3} \sigma_{4}) kVA \\
\|M_{a}\| &= 5 \sqrt{2} kVA \\
\|M_{b}\| &= 5 \sqrt{2} kVA \\
\|M_{l}\| &= 10 \sqrt{2} kVA
\end{align*}
\]
Note that the terms $M_a$ and $M_b$ in Equation (4.85) agree with the traditional definition of apparent power when each branch is examined separately as $\|S^a_t\| = \|S^b_t\| = 5\sqrt{2}kVA$. In this case the traditional definition does consider the two existing forms of power; however, this traditional definition also leads to the erroneous conclusion that $S_t = 10kVA$ is the total volt-amperes required by two branches. The base of the error lays on the fact that the traditional definition of apparent power adds $M_a$ and $M_b$ geometrically before adding common terms first.

4.2.5 Circuit Involving a HGL

When a circuit involves a Harmonic Generating Load (HGL) such as the circuit shown in Figure 4.8, a bidirectional flow of power occurs. This circuit is used with a two-fold purpose: firstly to describe how to apply the proposed technique to this type of circuit, and secondly to answer a thus far unresolved question presented by Czarnecki [45]. Only the first purpose will be covered in this Chapter while the other will be covered in the next Chapter.

![Resistive circuit with P=Q=0 at the cross section xx, yet at this cross section S≠0](image)

\begin{align*}
v(t) &= 100\sqrt{2}\sin(\omega t) \\
j(t) &= 100\sqrt{2}\sin(2\omega t)
\end{align*}

Figure 4.8 Resistive circuit with P=Q=0 at the cross section xx, yet at this cross section S≠0

The excitation sources and the time domain equation for this circuit are given by,
\[ v(t) = \sqrt{2}[100 \sin(\omega t)] \quad (4.86a) \]
\[ j(t) = 50\sqrt{2} \sin(2\omega t) \quad (4.86b) \]
\[ v(t) = 5i_s(t) - 4j(t) \quad (4.86c) \]

The signals \( \sqrt{2}[100 \sin(\omega t)] \) and \( 50\sqrt{2} \sin(2\omega t) \) in Equation (4.86) can be replaced by their RMS values in the geometric algebra \( G_3 \) domain by

\[ v(t) = \sqrt{2}[100 \sin(\omega t)] \rightarrow \langle V \rangle_1 = -100\sigma_2 \quad (4.87a) \]
\[ j(t) = 50\sqrt{2} \sin(2\omega t) \rightarrow \langle J \rangle_2 = 50\sigma_1 \sigma_3 \quad (4.87b) \]

Notice that there are two input signals each with one component; therefore, the output signal is composed by as many terms as terms in the input signals. Consequently if one input signal has \( m \) components and the other input signal has \( n \) components, then the output signal must have \( m + n \) components. Additionally, each component must rotate at \( k \) times the fundamental frequency where \( k \) denotes the grade of the \( k \)-vector which equals the input signal’s harmonic order. Thus,

\[ e^{-\frac{1}{2}\hat{\omega} t} \langle V \rangle_1 e^{\frac{1}{2}\hat{\omega} t} = 5\left[ e^{-\frac{1}{2}\hat{\omega} t} \langle I_s \rangle_1 e^{\frac{1}{2}\hat{\omega} t} + e^{-\frac{1}{2}2\hat{\omega} t} \langle I_s \rangle_2 e^{\frac{1}{2}2\hat{\omega} t} \right] - 4\left[ e^{-\frac{1}{2}2\hat{\omega} t} \langle J \rangle_2 e^{\frac{1}{2}2\hat{\omega} t} \right] \quad (4.88) \]

Note that in Equation (4.88) \( \langle I_s \rangle_1 \) and \( \langle I_s \rangle_2 \) denote the currents’ vector and bivector components respectively. Notice also that the vector components rotate at fundamental frequency \( \omega \) while the bivector components rotate at twice the fundamental frequency.

Analyzing the circuit at instant \( t = 0 \), the result yields,

\[ -100\sigma_2 = 5(\langle I_s \rangle_1 + \langle I_s \rangle_2) - 4[50\sigma_1 \sigma_3]. \quad (4.89) \]

Equation (4.89) is solved in the same way as any equation in the \( G_N \) domain.
4.3 Chapter Summary

This chapter reviews essential concepts of geometric algebra and introduces both: the new circuit analysis approach and the power multivector concept. The examples show the importance of the laws of operation and how the framework of geometric algebra overcomes the weaknesses encountered in the frequency-domain. *Additionally, the power multivector shows how the traditional definition of apparent power, given by the equation* \( S = ||V|| ||I|| \), *incorrectly performs a geometric addition before adding common terms first.* The examples also show that: in non-sinusoidal condition, Steinmetz’s circuit analysis framework provides only half or less of the information required to perform an energy analysis.
Chapter Five: Examination of the Results with Some Perspective using the Principle of Conservation of Energy and Kuhn’s Five Criteria

5.1 Introduction

The principle of conservation of energy is a well-known and well established empirical principle. In fact, it should be expected that any theory involving energy complies with this principle. As is shown in this chapter, the principle of conservation of energy is used to support the power theory presented in this dissertation because this principle is often used as a litmus test in power systems to test the validity of a proposed approach. If energy is not conserved in a given analysis, this would indicate an error in either the theory or the application of the theory. Although adherence to the principle of conservation provides a measure of the soundness of an energy-related theory; more criteria are required to assess the value of the proposed theory in relation to other well-established theories.

Thomas Kuhn [46] has suggested that a worthwhile scientific theory should satisfy five criteria, namely, the theory should be accurate, consistent, of broad scope, simple, and fruitful. Thus, the theory proposed in this dissertation is examined according to these two judging parameters, Kuhn’s five criteria and the principle of conservation of energy. While conservation is used to measure the soundness of the theory; Kuhn’s five criteria are used to assess the value of the proposed theory in relation to four other well-established power theories. Additionally, since a fair way to compare the virtues and the deficiencies of the different power theories is through numerical examples; the power properties of several circuits with non-sinusoidal sources and linear loads are examined using the theory of: Budeanu, Fryze, the IEEE standard 1459-2010, the CPC and the power theory presented in this dissertation. The rationale for selecting these four reference power theories over the many others is now briefly discussed. Budeanu’s model
is still the most well-known and accepted power theory today. Fryze’s approach is the second oldest power theory, the first one opposing Budeanu’s theory and the base of the CPC power theory. The IEEE standard 1459-2010 has the support of the IEEE, is the latest attempt to solve the century-old challenge of developing a power theory suitable for non-sinusoidal situations and is the result of a joint work of many scientists and engineers from all over the world with a number of them recognized as worldwide authorities such as Emmanuel. Lastly the CPC power theory is yet to be challenged [47]; is the most powerful tool available today and is the product of 25 years of work [3] of a worldwide authority honoured by the IEEE with the distinction of Fellow for his contributions to the solution of the unresolved, century-old problem of a power theory suitable for non-sinusoidal situations.

5.2 Fulfilment of the Principle of Conservation of Energy

The demonstration provided in this section is a modified version of the one published in [41, 42]. Although the design of compensators in non-sinusoidal situations can be performed in the same way they are designed under sinusoidal conditions, the algebra of complex numbers is ineffective for this task. In contrast, geometric algebra is very effective. Nevertheless, before proceeding with the demonstration let us examine first the process of designing a compensator in the frequency-domain under sinusoidal conditions.

\[ v(t) = \sqrt{2} \left[ Asin(\omega t) + Bcos(\omega t) + \sum_{j=2}^{d} D_j sin(j\omega t) + \sum_{j=2}^{e} E_j cos(j\omega t) \right] \]

Figure 5.1 Circuit used to show the frequency-domain’s inefficiency and the $G_N$ domain’s efficiency in non-sinusoidal circuit analysis.
In the sinusoidal case, the load’s current for the circuit of Figure 5.1a, is given by

\[ i_{load}(t) = Gv(t) + jBv(t) \] (5.1)

Equation (5.1) is a time-domain equation where the excitation \( v(t) \) is a voltage signal; the response \( i_{load}(t) \) is the current through the load, the scalar \( G \) is the conductance, the imaginary number \( jB \) represents the susceptance and \( G + jB = Y \) is the admittance. Since the admittance \( Y \) is composed of a scalar and a pure imaginary number then two operations are performed on the voltage signal \( v(t) \), i.e. \( Gv(t) \) and \( jBv(t) \). In the operation \( Gv(t) \) a dilatation occurs whenever \( G > 1 \) while a contraction happens if \( 0 < G < 1 \). In the operation \( jBv(t) \) a 90° rotation takes place due to the \( \pm j \) operator. The direction of the rotation is determined by the sign of the \( j \), while the magnitude of \( B \) determines whether a dilatation or a contraction is performed. The term \( jBv(t) \) in Equation (5.1) can be cancelled out by placing a second admittance in parallel with the load such that \( Y_{cp} = jB_{cp} = -jB \). Thus, the source current becomes,

\[ i_{scp}(t) = Gv(t) + jBv(t) + jB_{cp}v(t) = Gv(t) \] (5.2)

For circuits involving either non-sinusoidal inputs or harmonic generating loads or both, an expression similar to Equation (5.2) cannot be written in the frequency-domain. As demonstrated in Chapter Three and in [41] and [42], *in non-sinusoidal conditions the frequency-domain lacks a set of laws to operate the partial results attained at each harmonic analysis*. However, notice that it is the mathematical tool what impedes writing an equation similar to Equation (5.2) in the frequency-domain. Luckily, a similar expression to Equation (5.2) can be written in the \( G_N \) domain. Note that although the terms \( Bv(t) \) and \( B_{cp}v(t) \) of Equation (5.2) cannot be added when they are transformed to the frequency-domain; their addition in the \( G_N \) domain is straightforward. Let \( v(t) \) be the non-sinusoidal voltage source of any given circuit. Therefore,
\[
\frac{v(t)}{\sqrt{2}} = A \sin(\omega t) + B \cos(\omega t) + \sum_{j=2}^{d} D_j \sin(j \omega t) + \sum_{j=2}^{e} E_j \cos(j \omega t) \quad (5.3)
\]

As the dimension of the \( G_N \) domain is based on the highest harmonic present in the circuit then either \( N = d + 1 \), or \( N = e + 1 \). Applying Equation (4.29) to Equation (5.3) yields the expression of the voltage source in the \( G_N \) domain, thus,

\[
V = -A \sigma_2 + B \sigma_1 + \sum_{j=2}^{d} D_j \Lambda_{i=1}^{j+1} \Lambda_{i=2} \sigma_i + \sum_{j=2}^{e} E_j \Lambda_{i=2}^{j+1} \sigma_i \quad (5.4)
\]

Notice that the spinor \( Y_h = [ G_h + B_h (\omega, L, C) \sigma_1 \sigma_2 ] S \) is the admittance of the \( h^{th} \) harmonic in the \( G_N \) domain. A current component \( I_{||} \), parallel to the voltage source \( V \), results when each term in Equation (5.4) is multiplied on the left by the scalar part of \( Y_h \), namely \( G_h \). Similarly, a current component \( I_{\perp} \), perpendicular to the voltage source \( V \), results when each term in Equation (5.4) is multiplied on the left by the bivector part of \( Y_h \), namely \( B_h (\omega, L, C) \sigma_1 \sigma_2 \). Notice that when \( B_h (\omega, L, C) < 0 \) an additional \( 180^\circ \) rotation is performed as \( \sigma_1 \sigma_2 = -1 = j^2 \). Consequently, each term in Equation (5.4) suffers either a \( 90^\circ \) rotation or a \( 270^\circ \) rotation but in either case a \( 90^\circ \) angle results between current and voltage terms of the same harmonic. Also, in addition to the \( 90^\circ \) rotation each voltage term suffers a dilatation or a contraction depending on the value of \( B \), thus,

\[
I_s = I_l = I_{||} + I_{\perp} \quad (5.5)
\]

Where \( I_{||} \) and \( I_{\perp} \) are given by,

\[
I_{||} = -G_1 A \sigma_2 + G_1 B \sigma_1 + \sum_{j=2}^{d} G_j D_j \Lambda_{i=1}^{j+1} \Lambda_{i=2} \sigma_i + \sum_{j=2}^{e} G_j E_j \Lambda_{i=2}^{j+1} \sigma_i \quad (5.6)
\]

\[
I_{\perp} = -B_1 A \sigma_1 - B_1 B \sigma_2 - \sum_{j=2}^{d} B_j D_j \Lambda_{i=2}^{j+1} \sigma_i + \sum_{j=2}^{e} B_j E_j \Lambda_{i=2}^{j+1} \sigma_i \quad (5.7)
\]
Just as in the sinusoidal case, the current’s perpendicular component $I_{\perp}$ can be cancelled out by placing a second admittance of value $Y_{cp} = B_{cph} \sigma_1 \sigma_2 = -B_h \sigma_1 \sigma_2$ in parallel with the load. This second admittance provides the specifications for the compensator and the current through $Y_{cp}$ is given by,

$$I_{cp} = -B_{cp1} A \sigma_1 - B_{cp1} B \sigma_2 - \sum_{j=2}^{d} B_{cpj} D_j \Lambda_{i=2}^{j+1} \sigma_i + \sum_{j=2}^{e} B_{cpj} E_j \Lambda_{i=1}^{j+1} \sigma_i \quad (5.8)$$

Applying Kirchhoff’s currents law yields the current through the supply source given now by $I_{scp} = I_{\parallel} + I_{\perp}$, as $I_{\perp} = -I_{cp}$. Notice that the specifications for the compensator given by $Y_{cp} = B_{cph} \sigma_1 \sigma_2 = -B_h \sigma_1 \sigma_2$ is exactly the same conclusion attained by Czarnecki [47] for the design of the compensator; however, an advantage over the CPC power theory appears immediately. Notice that depending on the topology of the compensator it is always possible to develop a set of $n$ equations and $n$ unknowns and to modify the amount of each current component to simplify the compensator. More importantly yet, notice that the definition of the reactive current in the CPC power theory [48] is directly obtained from the magnitude of Equation (5.8) [41, 42].

$$\|I_{cp}\| = \sqrt{\sum_{j} B_{cpj}^2 \|U_j\|^2} = \|I_{r}\|_{CPC} \quad (5.9)$$

The current through the supply source is now governed by,

$$I_{scp} = I_{cp} + I_{\parallel} + I_{\perp} \quad (5.10)$$

Left multiplication by the Equation (5.4) over each current term in Equation (5.10) yields the total volt-amperes at the source, the compensator and the load. The source’s multivector power is denoted as $M_{scp}$, the compensator’s multivector power is denoted as $M_{cp}$, and the load’s multivector power is given by $M_{\parallel} + M_{\perp}$. Thus,
Notice that in this equation the total volt-amperes at the input terminal is equal to the sum of the volt-amperes at each load component; thus, this power theory has the most solid foundation as it fulfills the principle of conservation of energy (PoCoE).

Using Kuhn’s terminology, the power theory presented in this research is externally consistent with the principle of conservation of energy. Its simplicity can be judged by recognizing that the total volt-amperes is defined by the geometric product of multivectors $V$ and $I$. Note that with the compensator in place the total volt-amperes at the supply source reduces to $M_{scp} = M_{||}$ since $I_{cp} = -I_{\perp}$ and consequently $M_{cp} = -M_{\perp} = -CN_r$. At this point it is important to examine the different components of $M_{||}$ and $M_{\perp}$; however, it is worth sorting them first as,

$$M_{||} = \langle M_{||} \rangle_0 + \sum_{i=1}^{N} \langle M_{||} \rangle_i$$

$$M_{\perp} = -CN_r = CN_{r(ps)} + CN_{r(hl)}$$

The term $\langle M_{||} \rangle_0$ is always a scalar and represents the active average power $P = \langle M_{||} \rangle_0$. This term is obtained by multiplying terms of the same harmonic order among multivectos $V$ and $I_{||}$. Therefore, as the partial results involve the square of $k$-vectors, the application of the correction factor given by Equation (4.68) is required before adding all the partial results. As stated in Chapter Four, this correction factor is necessary because $k$-vectors do not always square to $+1$. The term $\sum_{i=1}^{N} \langle M_{||} \rangle_i = M_{||} - \langle M_{||} \rangle_0 = CN_d$ represents the new term denoted as degrading power and involves the summation of cross-frequency products between multivectos $V$ and $I_{||}$. This last term...
sometimes coincides with the *scattered power* term of the CPC power theory; however, determining when and why these two terms coincide is outside of the scope of this dissertation. The term $CN_{r(ps)}$ involves products of the same frequency between multivectors $V$ and $I_{\text{ll}}$ and represents the *reactive power* due to the phase-shift among voltage and current. Finally, the term $CN_{r(hl)}$ represents the new quantity called *reactive power* due to harmonic interactions and involves cross-frequency products among $V$ and $I_{\text{ll}}$. The discovery of this new form of reactive power, reported for the first time in [41] provides evidence of the fruitfulness of the power theory presented in this dissertation. The total volt-amperes at the load can be finally decomposed as,

$$ M_I = P + CN_{r(ps)} + CN_{r(hl)} + CN_d $$  \hspace{1cm} (5.14)

Since the bivector $\sigma_1\sigma_2$ performs a $90^\circ$ rotation similar to the operator $j$ in the complex numbers algebra and as $CN_{r(hl)} = CN_d = 0$ and $\|CN_{r(ps)}\| = Q$ in the sinusoidal case, one can see that Equation (5.14) reduces to the well-known equation $S = P + jQ$; then,

$$ M_I = P + CN_{r(ps)}\sigma_1\sigma_2 = P + jQ, \quad \|M_I\| = \|S\| $$  \hspace{1cm} (5.15)

Equation (4.67), i.e. $M = VI$, provides a unified power equation for sinusoidal and non-sinusoidal situations; which demonstrate the fruitfulness and the broadness in scope of the power theory presented in this dissertation. Further evidence of these two of Kuhn’s criteria is seen when the traditional definition of apparent power is contrasted with the definition provided here. Notice that although the two definitions are identical in the sinusoidal case as Equation (4.66) and (5.15) show, in general $\|M\| \neq \|V\|\|I\|$; therefore, the traditional definition of apparent power given by $S = \|V\|\|I\|$ cannot be mathematically justified in the $G_N$ domain for the following four reasons:

1. It violates the established definition of the norm given by $\|M\| = \sqrt{\langle MM \rangle_0}$ – Equation (4.67).
It breaks the mathematical coherence of the theory,

\[
\| \mathbf{M} \| = \sqrt{\langle \mathbf{M} \mathbf{M} \rangle_0} = \sqrt{P^2 + CN^2_{r(ps)} + CN^2_{r(hl)} + CN^2_d} \neq ||V|| ||I||
\]

Its application leads to a violation of the principle of conservation of energy. In contrast, the net apparent power concept (NAP) presented here takes into account the net flow of all the power components present in the element or circuit section [42].

5.3 Comparative Analysis of Five Power Theories

The label 5.2SOFF identifies the circuit in Figure 5.2 when the switch \( S \) is opened while the label 5.2SON identifies the resulting circuit when the switch is closed. The \( LC \) elements on the switch’s branch are referred to as the \( LCb \) —branch to differentiate from the load’s \( LC \) —branch. As the analysis of circuits 5.2SOFF and 5.2SON provides the baseline for all the power theories, the analysis is performed in the frequency-domain and the \( G_N \) domain and the two results are compared. Subsequently, the power properties of circuits 5.2SOFF and 5.2SON are examined first by the four power theories discussed in Chapter Two, i.e. Budeanu, Fryze, The IEEE Standard 1459-2010 and the CPC, and subsequently by the power theory proposed in this dissertation which can be identified as the \( G_N \) domain power theory. This allows viewing and discussing the contributions and deficiencies of every power theory in relation to Kuhn’s five criteria and the Principles of Conservation of Energy. The rationale for this approach is twofold; firstly, to further demonstrate the necessity of the paradigm-shift in non-sinusoidal circuit analysis, and secondly, to further demonstrate that the challenge of the appropriate interpretation of the power phenomena in non-sinusoidal conditions is rooted on the limitations of Steinmetz’s circuit analysis framework, here referred to as Steinmetz’s technique. Equation (5.16) provides the excitation source for the circuit of Figure 5.2, thus,
As demonstrated in Chapter Three, a time-domain signal composed by the addition of sinusoidal signals of different frequency does not have a representation in the frequency-domain. Thus, the frequency-domain requires an analysis be made for each harmonic. Equation (5.17) provides the results of this fractionalized circuit analysis for circuit 5.2SOFF while Figure 5.3 provides a graphical view of the results. The super index in each quantity of Equations (5.17) associates the quantity with the circuit branch while the sub-index associates the quantity with the harmonic, e.g. \( I_1^S = -j10A \) describes the source’s current at fundamental frequency. The frequency-domain analysis yields,

\[
\begin{align*}
\text{Analysis at } \omega &= \omega_1 \\
V_1^S &= -j100V \\
I_{1R}^{\text{RLC}} &= 30 - j10A \\
I_1^L &= 30A \\
I_1^S &= -j10A \\
S_1^* &= V_1^S I_1^* = (1 + j0)kVA \\
P_1^S &= 1kW \\
Q_1^S &= j0kVAR
\end{align*}
\]

\[
\begin{align*}
\text{Analysis at } \omega &= 3\omega_1 \\
V_3^S &= -j100V \\
I_{3R}^{\text{RLC}} &= -30 - j90A \\
I_3^L &= 90A \\
I_3^S &= -30 - j90A \\
S_3^* &= V_3^S I_3^* = (9 + j3)kVA \\
P_3^S &= 9kW \\
Q_3^S &= j3kVAR
\end{align*}
\]
Although the total active power $P_t$ can be found by adding $P_1^S$ and $P_3^S$ in Equation (5.17), or more generally through the expression $P_t = \sum_{i=1}^{n} P_i$, it must be remembered that: there are no laws of operation among the quantities on the two columns of Equation (5.17) and that the total power equation, i.e. $P_t = \sum_{i=1}^{n} P_i$, is an equation that belongs exclusively to the time-domain. Additionally, as discussed in Chapter Three, the equations that determine the magnitude of the voltage, i.e. $\|V\| = \sum_{h=1}^{n} \sqrt{V_h^2}$, and the current, i.e. $\|I\| = \sum_{h=1}^{n} \sqrt{I_h^2}$ are also equations that belong exclusively to the time-domain. Consequently, although the scientific community accepts that the traditional definition of apparent power, i.e. $S = \|V\|\|I\|$, is a frequency-domain quantity; from the rigorous mathematical perspective, it is more appropriate to consider the non-sinusoidal apparent power $S$ a time-domain quantity as its two constituents, i.e. $\|V\|$ and $\|I\|$, are time-domain quantities. Emmanuel, Czarnecki and many other scientist view the non-sinusoidal apparent power $S$ as a norm; however, from the rigorous mathematical perspective, a norm cannot be defined unless a clear and unambiguous set of laws of operation among all the elements of the set, e.g. all the elements in Equation (5.17), are in place first [41, 42]. Therefore, as in non-sinusoidal situations the frequency-domain does not provide a set of laws of operations among the results attained for each harmonic analysis then it is incorrect to view the apparent power $S$ as a norm.
Analysis of Figure 5.3(a) shows that at fundamental frequency, i.e. at $\omega_1$, the $LC$-branch acts as compensator for the $RLC$-branch and releases the source from having a current component that is out-of-phase with the voltage. Thus, as the current and voltage are in-phase, the source’s reactive power is zero. However, at threefold fundamental frequency, i.e. at $3\omega_1$, the $LC$-branch acts as an open circuit and subsequently a current component that is out-of-phase with the voltage appears at the source and therefore the source’s reactive power is no longer zero. As at $3\omega_1$ the $LC$-branch is not able to meet the $RLC$-branch’s reactive power requirements, the source is forced to meet the $RLC$-branch’s active and the reactive power requirements at $3\omega_1$. Any power theory needs to presents these facts at all times.

Figure 5.4 and Equation (5.18) provide the analysis of circuit 5.2SON. In this case, the added $LCb$-branch to the circuit’s load affects only the source’s current and only at $3\omega_1$ as the current at $\omega_1$ on the $LCb$-branch is zero, i.e. $I_1^{LCb} = 0A$; therefore, $I_3^{sb} = -j90A$ reflects the change in the source’s current at $3\omega_1$ when the switch in Figure 5.2 is closed. The super index in the quantities of Equation (5.18) denotes the circuit branch, while the sub index denotes the harmonic order, i.e. $V_1^S$ describes the voltage at fundamental frequency on the source branch, while $I_3^{RLC}$ describes the current at threefold fundamental frequency on the $RLC$ branch. This convention applies to the remaining equations in the chapter unless explained otherwise.

\[
\begin{align*}
\text{Analysis at } \omega = \omega_1 & \quad \text{Analysis at } \omega = 3\omega_1 \\
V_1^S &= -j100V & V_3^S &= -j100V \\
I_1^{RLC} &= 30 - j10A & I_3^{RLC} &= 30 - j90A \\
I_1^{LC} &= 30A & I_3^{LC} &= 0A \\
I_1^{LC2} &= 0A & I_3^{LC2} &= 30A \\
I_3^S &= -j10A & I_3^{S2} &= -j90A \\
S_1^S &= V_1^S I_1^S = (1 + j0)kVA & S_3^{S2} &= V_3^S I_3^{S2} = (9 + j0)kVA \\
P_1^S &= 1kW & P_3^{S2} &= 9kW \\
Q_1^S &= j0kVAR & Q_3^{S2} &= j0kVAR \\
\end{align*}
\]
Analysis of Figure 5.4 shows that at $3\omega_1$ the $LC_b$-branch acts as compensator for the circuit’s load and releases the source from having a current component that is out-of-phase with the voltage. Thus, as the current and voltage are in-phase, the source’s reactive power is zero. Notice that the $LC_b$-branch acts as an open circuit at $\omega_1$. As the $RLC$-branch behaves as an $RC$ circuit at $\omega_1$ but as an $RL$ circuit at $3\omega_1$; its reactive power requirements are a function of the frequency. Thus, while the existing $LC$-branch compensates the $RLC$-branch’s reactive power requirements at $\omega_1$, the $LC_b$-branch compensates the $RLC$-branch’s reactive power requirements at $3\omega_1$. Consequently, removal of both $LC - branches$ forces the source to meet all the $RLC - branch's$ total volt-ampere requirements at both frequencies. Any power theory must deliver all the above facts.
5.3.2 Analysis of the Power Phenomena According to Budeanu’s Power Theory

Equations (5.19) and (5.20) provide the power quantities of circuit 5.2SOFF using Budeanu’s power theory. In these equations \( \|V^S\| = 100\sqrt{2}V, \|I^S\| = 10\sqrt{2}A, \|I^{RLC}\| = 100A, \|I^{LC}\| = 30A \) and \( V_1^{RLC} = V_3^{RLC} = V_1^{LC} = V_3^{LC} = V_1^S = 100V \), thus,

\[
\begin{align*}
S_1^S &= V_1^S I_1^S = (1 + j0)kVA \\
S_3^S &= V_3^S I_3^S = (9 + j3)kVA \\
S^S &= \|V^S\|\|I^S\| = \sqrt{182}kVA \\
P^S &= (1 + 9) = 10kW \\
Q_B^S &= j[0 + 3]kVAR = j3kVAR \\
D_B^S &= \sqrt{73} \approx 8.54kVA
\end{align*}
\]

\[
\begin{align*}
S_1^{RLC} &= V_1^{RLC} I_1^{RLC} = (1 + j3)kVA \\
S_3^{RLC} &= V_3^{RLC} I_3^{RLC} = (9 + j3)kVA \\
S^{RLC} &= \|V^{RLC}\|\|I^{RLC}\| = 10\sqrt{2}kVA \\
P^{RLC} &= (1 + 9) = 10kW \\
Q_B^{RLC} &= j[3 + 3]kVAR = j6kVAR \\
D_B^{RLC} &= 8kVA
\end{align*}
\]

\[
\begin{align*}
S_1^{LC} &= V_1^{LC} I_1^{LC} = -j3kVA \\
S_3^{LC} &= V_3^{LC} I_3^{LC} = 0kVA \\
S^{LC} &= \|V^{LC}\|\|I^{LC}\| = 3\sqrt{2}kVA \\
P^{LC} &= 0kW \\
Q_B^{LC} &= j[0 - 3]kVAR = -j3kVAR \\
D_B^{LC} &= 3kVA
\end{align*}
\]

Equation (5.20) shows that the LC-branch’s reactive power of 3kVAR is insufficient for compensating the RLC-branch’s reactive power which is 6kVAR and therefore the source needs to compensate the RLC-branch’s remaining reactive power requirements of 3kVAR. Examination of Equation (5.19) confirms that the source’s reactive power is \( Q_B^S = j3kVAR \) which is in agreement with the frequency-domain analysis of circuit 5.2SOFF. Thus, Equation (5.19) confirms that the source’s reactive power appears as a consequence of the LC-branch’s inability to compensate all the RLC-branch’s reactive power requirements.
Budeanu’s power theory is simple; the active power is estimated as \( P_t = \sum_{i=1}^{n} P_i \), the reactive power as \( Q_t = \sum_{i=1}^{n} Q_i \) and the distortion power as \( D_B^S = \sqrt{S^2 - P^2 - Q_B^R} \).

However, Budeanu assumed erroneously that in non-sinusoidal situations, the operation of addition is defined in the frequency-domain which is not. Without acknowledging that the frequency-domain does not provide a set of laws of operations for \( Q_1, Q_2, Q_3 \ldots Q_n \), Budeanu defined the total reactive power as the arithmetic addition of the reactive power at each harmonic, i.e. \( Q_B^S = \sum_{h=1}^{n} Q_h \). Thus, for circuit 5.2SOFF the reactive power is estimated as \( Q_B^S = Q_1 + Q_3 = j3kVAR \). Although Budeanu’s power theory has been severely criticised by Czarnecki \[51\] for its lack of connection with the physical phenomena; this disconnection should not be surprising as the definition of reactive power given by \( Q_B^S = \sum_{h=1}^{n} Q_h \) cannot be supported from the rigorous mathematical perspective. Budeanu’s power theory was also criticized by Waldo V. Lyon \[49\] as early as 1935; however, although Lyon’s circuit in \[49\] shows the internal inconsistency of Budeanu’s power theory, the fact that the theory does not comply with the principle of conservation of energy, should be sufficient to disqualify Budeanu’s power theory. According to the principle of conservation of energy, in steady-state operation, the total volt-ampere at the source must equal the total volt-ampere at the load; therefore, consistency with the principle of conservation of energy requires that in Budeanu’s power theory the following three equations, \( P^S = P^{LC} + P^{RLC} \), \( Q_B^S = Q_B^{LC} + Q_B^{RLC} \) and \( D_B^S = D_B^{LC} + D_B^{RLC} \) must hold true. Unfortunately, although the equations for the active average power and the reactive power comply with the principle of conservation of energy, the equation for the distortion power does not. Inconsistency with the Principle of Conservation of Energy (PoCoE) yields inconsistency on the interpretation of the power at the source and at the load. Note that according to Budeanu’s power theory, from the source’s perspective, the load requires 10kW of active average power, 3kVAR of reactive power and 8.54kVA of distortion power as per interpretation of Equation (5.19); unfortunately, the same conclusion cannot be reached from interpreting the results at the load as it is uncertain how to interpret the distortion power quantity on Equation (5.20) but in any case neither the addition nor the subtraction of \( D_B^{LC} = 3kVA \) and \( D_B^{RLC} = \).
$8kVA$ leads to $D_B^S = 8.54kVA$ i.e. $D_B^{LC} + D_B^{RCL} = 11kVA$ and $D_B^{RCL} - D_B^{LC} = 5kVA$. Consequently, as the distortion power $D$ is not conservative then the PoCoE cannot be satisfied using Budeanu’s power theory. Also, Budeanu’s power theory does not provide any provisions on how to design a compensator, nor does the theory provide an explanation on the physical meaning of the distortion power $D$. As circuit 5.2SON showed previously, the circuit’s reactive power can be entirely compensated with the $LCb$ –branch. Equations (5.20) to (5.22) show the power quantities of circuit 5.2SON according to Budeanu’s power theory.

\[
\begin{align*}
S_{1Cb}^{LCb} &= V_1^{LCb} I_1^{LCb} = 0kVA \\
S_{3Cb}^{LCb} &= V_3^{LCb} I_3^{LCb} = -j3kVA \\
S_{LCb}^{LCb} &= ||V_{LCb}^{LCb}|| ||I_{LCb}^{LCb}|| = 3\sqrt{2}kVA \\
P_{LCb}^{LCb} &= 0kW \\
Q_{B}^{LCb} &= j[0 - 3]kVAR = -j3kVAR \\
D_{B}^{LCb} &= 3kVA
\end{align*}
\] (5.21a)

\[
\begin{align*}
S_{1b}^{SBb} &= V_1^{SBb} I_1^{SBb} = (1 + j0)kVA \\
S_{3b}^{SBb} &= V_3^{SBb} I_3^{SBb} = (9 + j0)kVA \\
S_{b}^{SBb} &= ||V_{SBb}^{SBb}|| ||I_{SBb}^{SBb}|| = 2\sqrt{41}kVA \\
P_{b}^{SBb} &= (1 + 9) = 10kW \\
Q_{B}^{SBb} &= j[0 + 0]kVAR = j0kVAR \\
D_{B}^{SBb} &= \sqrt{64} = 8kVA
\end{align*}
\] (5.22a)

Examination of Equations (5.20) to (5.21) and comparison of Equations (5.19) and (5.22) shows that, at $3\omega_1$ the $LCb$-branch acts as a compensator for the circuit’s load and releases the source from having a current component that is out-of-phase with the voltage. Thus, at $3\omega_1$ the current and voltage are in-phase also and consequently the source’s reactive power is now zero at $\omega_1$ and $3\omega_1$, which is in agreement with the frequency-domain analysis of circuit 5.2SON. Moreover, examination of Equations (5.20) to (5.22) leads to conclude that the $LCb$ –branch compensates also the distortion power of the load’s $LC$ – branch such that the source’s distortion power equals the distortion power on the $RLC$ –branch. Unfortunately, this conclusion yields an internal
inconsistency in the theory for two reasons: firstly, because the same conclusion cannot be attained from interpreting the power phenomena in circuit 5.2SOFF, i.e., by interpreting Equations (5.19) and (5.20); secondly, because the reduction in distortion power at the load is 3kVA while the source’s distortion power reduction is only 0.54kVA, as $D_B^s \approx 8.54kVA$ and $D_B^{sh} = 8kVA$. There is a disconnection between the distortion power at the source and the distortion power at the load in Budeanu’s power theory. This inconsistency in the interpretation of the distortion power quantity $D_B$ makes Budeanu’s power theory internally inconsistent.

Although Budeanu’s power theory is still used by a good number of researchers, e.g. [30], [33], [50] and many countries today still use revenue meters that have his theory embedded on them; the theory is internally inconsistent, is inconsistent with the PoCoE and other flaws have been irrefutably demonstrated by Czarnecki more than 25 years ago [51].

### 5.3.3 Analysis of the Power Phenomena According to Fryze’s Power Theory

Equations (5.23) to (5.25) provide the power quantities of circuit 5.2SOFF according to Fryze’s power theory. In these equations $v^s$ and $i^s$ are time-domain quantities, thus,

\[
\begin{align*}
    v^s &= 100\sqrt{2}(\sin\omega t + \sin 3\omega t)V \\
    i^s &= \sqrt{2}[10 \sin\omega t + 30 \cos 3\omega t + 90\sin 3\omega t]A \\
    p^s &= \|v^s\|\|i_a^s\| = 10kW \\
    i_a^s &= \frac{P}{\|v^s\|^2}v = 50\sqrt{2}(\sin\omega t + \sin 3\omega t)A \\
    i_{r_F}^s &= i^s - i_a^s = \sqrt{2}[−40 \sin\omega t + 30 \cos 3\omega t + 40\sin 3\omega t] \quad \|i_{r_F}^s\| = 10\sqrt{41}A \\
    s^s &= \|v^s\|\|i^s\| = \sqrt{182kVA} \\
    q_{r_F}^s &= \|v^s\|\|i_{r_F}^s\| = \sqrt{s^s^2 - p^s^2} = \sqrt{82} \approx 9.06kVAr
\end{align*}
\]
Fryze’s power theory is defined in the time domain where all the operations are perfectly defined. Thus, from the mathematical perspective, Fryze’s power theory provides a better mathematical formality. Fryze defines only two types of current, active and reactive. It must be highlighted that since the definition of the reactive current is based on the definition of the active current, a failure to accurately estimate the active current yields a miscalculation of the reactive power as \( Q_F = ||v|| ||i_{rF}|| = ||v|| ||i^S - i_i^S|| \).

Notice how different is the magnitude of the reactive power using Fryze’s definition and Budeanu’s definition, i.e. \( Q_F^S = 3kVA r \) while \( Q_F^B \approx 9kVA r \). Certainly, this affects both, engineers and consumers; while the uncertainty on which value should be used to design the compensator affects engineers, customers are affected as their bills for power factor penalty does not depend on the actual consumption of reactive power but on the definition embedded in their metering device. Such a case is reported by Berrisford [52] where a change in the customer’s meter performed by the utility company, BC Hydro, yielded a higher bill although the customer’s load had not changed. The investigation
revealed that while the old meter had embedded Budeanu’s power theory, the replacement meter had Fryze’s power theory.

Fryze’s power theory is the simplest among the four power theories under examination; it requires estimating only two power quantities, the active average power \( P \) and the reactive power \( Q_F \). Examination of Equation (5.25) confirms that the \( LC - branch \)’s reactive current is \( \|i_{rF}^{LC}\| = 30A \); however, in contrast to Budeanu’s power theory the reactive power is estimated as \( Q_F^{LC} = 3\sqrt{2}kVAR \), i.e. a value \( \sqrt{2} \) times higher.

Fryze’s power theory has been criticised by Czarnecki [48] because the postulated definition of reactive power is not directly related to the load and therefore it does not accurately describe the physical phenomena; however, failure to comply with the PoCoE should be sufficient to disqualify Fryze’s power theory. Moreover, this theory is inconsistent with the balance principle of the reactive power. Notice that neither the addition nor the subtraction of \( Q_F^{RLC} \) and \( Q_F^{LC} \) yields \( Q_F^S \) i.e. \( Q_F^S \approx 9.1kVA \neq Q_F^{RLC} + Q_F^{LC} \approx 14.24kVA \) and \( Q_F^S \neq Q_F^{RLC} - Q_F^{LC} \approx 5.8kVA \). Thus, as the reactive power \( Q_F \) is not conservative then Fryze’s power theory does not satisfy the PoCoE. More importantly, Fryze’s power theory does not provide any provisions on how to design a compensator. As circuit 5.2SON showed previously, the circuit’s reactive power can be entirely compensated by adding another \( LC \)-branch. Equations (5.24) to (5.27) provide the power quantities of circuit 5.2SON according to Fryze’s power theory.

\[
\begin{align*}
\nu^{LCb} &= 100\sqrt{2}(\sin \omega t + \sin 3\omega t)V = \nu^S \\
i^{LCb} &= 30\sqrt{2}[\sin 3\omega t]A \\
P^{LCb} &= 0kW \\
\|i_{a}^{LCb}\| &= \|v^{LCb}\|, n^{LCb} = 0A \\
i_{rF}^{LC} &= i^{LCb} - i_{a}^{LCb} = 30\sqrt{2}[\sin 3\omega t]A \\
S^{LCb} &= \|v^{LCb}\| \|i^{LCb}\| = 3\sqrt{2}kVA \\
Q_F^{LC} &= \|v^{LCb}\| \|i_{rF}^{LC}\| = \sqrt{S^{LCb^2} - P^{LCb^2}} = 3\sqrt{2}kVAR
\end{align*}
\]
\begin{align*}
\|v_{sb}\| &= 100\sqrt{2}(\sin \omega t + \sin 3\omega t)V \\
\|i_{sb}\| &= 100\sqrt{2}V \\
(5.27a) \\
\|i_{sb}\| &= 10\sqrt{2}A \\
(5.27b) \\
P_{sb} &= 10\, \text{kW} \\
(5.27c) \\
\|i_{a}^{Sb}\| &= \frac{P_{sb}}{\|v_{sb}\|^2} v_{sb} = 50\sqrt{2}(\sin \omega t + \sin 3\omega t)A \\
\|i_{a}^{Sb}\| &= 50\sqrt{2}A \\
(5.27d) \\
\|i_{r}^{Sb}\| &= i_{sb} - i_{a}^{Sb} = 10\sqrt{2}[-4 \sin \omega t + 4\sin 3\omega t] \\
\|i_{r}^{Sb}\| &= 40\sqrt{2}A \\
(5.27e) \\
S_{sb} &= \|v_{sb}\|\|i_{sb}\| = 2\sqrt{41}kVA \approx 12.806\, \text{kVAR} \\
(5.27f) \\
Q_{F}^{Sb} &= \|v_{sb}\|\|i_{r}^{Sb}\| = \sqrt{S_{sb}^2 - P_{sb}^2} = 8\, \text{kVAR} \\
(5.27g)
\end{align*}

Comparison of Equations (5.23) and (5.27) shows that, contrary to the results attained with Budeanu’s power theory, the LCb-branch does not release the source from having a current component that is out-of-phase with the voltage; according to Fryze’s power theory, the LCb-branch causes a reduction only. Since the source’s reactive current reduces from $\|i_{r}^{S}\| = 10\sqrt{41} \approx 64.03A$ to $\|i_{r}^{Sb}\| = 40\sqrt{2} \approx 56.57A$ then the source’s reactive power reduces from $Q_{F}^{S} \approx 9.06kVAR$ to $Q_{F}^{Sb} = 8kVAR$ which implies that the LCb-branch does not eliminate completely the load’s reactive power requirements. This interpretation is unacceptable because the LCb-branch also causes the source current $i_{sb}$ to become in-phase with the voltage source $v_{sb}$ as Equation (5.27) shows, i.e. $v_{sb} = 100\sqrt{2}(\sin \omega t + \sin 3\omega t)V$ and $i_{sb} = 10\sqrt{2}[\sin \omega t + 9\sin 3\omega t]A$. Consequently, as predicted by Budeanu’s power theory, the LCb-branch does eliminate the load’s reactive power requirements. The fact that Fryze’s power theory yields a nonzero value of reactive power when voltage and current are in-phase leads to conclude that Fryze’s power theory is internally inconsistent. In Fryze’s power theory, the disconnection is between the reactive power at the source and the reactive power at the load. Thus, Fryze’s power theory does not satisfy the PoCoE.

Although Fryze’s power theory is embedded in revenue meters approved in a number of countries, i.e. Canada; the theory is internally inconsistent, is inconsistent with the PoCoE, the balance principle of the reactive power and other flaws have been demonstrated by Czarnecki [48] and other authors, i.e. [41].
5.3.4 Analysis of the Power Phenomena According to the IEEE Standard 1459-2010

Equations (5.28) to (5.30), where $V_i^S$ and $I_i^S$ describe RMS values, provide the power quantities of circuit 5.2SOFF according to the IEEE standard.

\[
\begin{align*}
\text{Fundamental power terms} & \quad \text{Nonfundamental power terms} \\
S_i^S &= V_i^S I_i^S = 1 \text{kVA} & S^S &= ||V^S|| ||I^S|| = 10\sqrt{182} \text{kVA} \quad (5.28a) \\
P_i^S &= V_i^S I_i^S \cos \phi_i^S = 1 \text{kW} & P^S &= \sum_{h=1}^{n} V_h I_h \cos \phi_h = 10 \text{kW} \quad (5.28b) \\
Q_i^S &= V_i^S I_i^S \sin \phi_i^S = 0 \text{kVAR} & D_i^S &= V_i^S I_i^S = 3\sqrt{10} \text{kVA} \quad (5.28c) \\
\phi_i^S &= 0 & D_V^S &= V_H^S I_H^S = \sqrt{2} \text{kVA} \quad (5.28d) \\
S_H^S &= V_H^S I_H^S = 6\sqrt{5} \text{kVA} & S_H^S &= V_H^S I_H^S = 3\sqrt{10} \text{kVA} \quad (5.28e) \\
N^S &= \sqrt{(S^S)^2 - (P^S)^2} = 10\sqrt{181} \text{kVA} & N^S &= \sqrt{(S^S)^2 - (P^S)^2} = 3\sqrt{2} \text{kVA} \quad (5.28f)
\end{align*}
\]

\[
\begin{align*}
\text{Fundamental power terms} & \quad \text{Nonfundamental power terms} \\
S_1^{RLC} &= V_1^{RLC} I_1^{RLC} = \sqrt{10} \text{kVA} & S^{RLC} &= ||V^{RLC}|| ||I^{RLC}|| = 10\sqrt{2} \text{kVA} \quad (5.29a) \\
P_1^{RLC} &= V_1^{RLC} I_1^{RLC} \cos \phi_1^{RLC} = 1 \text{kW} & P^{RLC} &= \sum_{h=1}^{n} V_h I_h \cos \phi_h = 10 \text{kW} \quad (5.29b) \\
Q_1^{RLC} &= V_1^{RLC} I_1^{RLC} \sin \phi_1^{RLC} = -3 \text{kVAR} & D_i^{RLC} &= V_1^{RLC} I_1^{RLC} = 3\sqrt{10} \text{kVA} \quad (5.29c) \\
\phi_1^{RLC} &= -71.56^\circ & D_V^{RLC} &= V_H^{RLC} I_H^{RLC} = \sqrt{10} \text{kVA} \quad (5.29d) \\
S_H^{RLC} &= V_H^{RLC} I_H^{RLC} = 3\sqrt{10} \text{kVA} & S_H^{RLC} &= V_H^{RLC} I_H^{RLC} = 3\sqrt{10} \text{kVA} \quad (5.29e) \\
N^S &= \sqrt{(S^{RLC})^2 - (P^{RLC})^2} = 10 \text{kVA} & N^S &= \sqrt{(S^{RLC})^2 - (P^{RLC})^2} = 3\sqrt{2} \text{kVA} \quad (5.29f)
\end{align*}
\]

\[
\begin{align*}
\text{Fundamental power terms} & \quad \text{Nonfundamental power terms} \\
S_1^{LC} &= V_1^{LC} I_1^{LC} = -3 \text{kVA} & S^{LC} &= ||V^{LC}|| ||I^{LC}|| = 3\sqrt{2} \text{kVA} \quad (5.30a) \\
P_1^{LC} &= V_1^{LC} I_1^{LC} \cos \phi_1^{LC} = 0 \text{kW} & P^{LC} &= \sum_{h=1}^{n} V_h I_h \cos \phi_h = 0 \text{kW} \quad (5.30b) \\
Q_1^{LC} &= V_1^{LC} I_1^{LC} \sin \phi_1^{LC} = 3 \text{kVAR} & D_i^{LC} &= V_1^{LC} I_1^{LC} = 0 \text{kVA} \quad (5.30c) \\
\phi_1^{LC} &= 90^\circ & D_V^{LC} &= V_H^{LC} I_H^{LC} = 3 \text{kVA} \quad (5.30d) \\
S_H^{LC} &= V_H^{LC} I_H^{LC} = 3 \text{kVA} & S_H^{LC} &= V_H^{LC} I_H^{LC} = 3 \text{kVA} \quad (5.30e) \\
N^S &= \sqrt{(S^{LC})^2 - (P^{LC})^2} = 3\sqrt{2} \text{kVA} & N^S &= \sqrt{(S^{LC})^2 - (P^{LC})^2} = 3\sqrt{2} \text{kVA} \quad (5.30f)
\end{align*}
\]
Notice how different is the magnitude of the reactive power using the IEEE’s definition \( Q_1^S \approx 0 \text{ kVAR} \), Budeanu’s definition \( Q_B^S = 3 \text{ kVAR} \), and Fryze’s definition \( Q_F^S \approx 9 \text{ kVAR} \). Again, on which value do engineers base the design of the compensator and on which base are the consumers billed, -Budeanu, Fryze or the IEEE standard? Although the IEEE’s fundamental power quantities fulfill the PoCoE the non-fundamental power quantities do not. Consequently, the IEEE standard is inconsistent with the PoCoE. More importantly, the standard does not provide any provisions on how to design a compensator, nor does the theory provide an explanation on the physical meaning of its quantities.

Notice that in circuit 5.2SOFF the fundamental frequency current and voltage are in-phase, so \( Q_1 = V_1 I_1 \sin 0 = 0 \); thus, according to the IEEE Standard the load’s reactive power is zero. This conclusion is unacceptable because circuit 5.2SON shows the load’s reactive power can be entirely compensated, but according to the IEEE standard the load’s reactive power is already zero which leads to conclude that the \( LCb \) - branch has no effect on the circuit’s reactive power. The contradiction in this example shows that the IEEE standard is internally inconsistent. By giving a zero value for the reactive power, i.e. \( Q_1 = 0 \), the IEEE standard suggests the load’s reactive power requirements have been fulfilled, yet the added \( LCb \) - branch proves the opposite. The IEEE standard does not explain how the \( LCb \)-branch, decreases the apparent power. Equations (5.29) to (5.32) show the power quantities of circuit 5.2SON according to IEEE’s power theory.

<table>
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<tr>
<th>Fundamental power terms</th>
<th>Nonfundamental power terms</th>
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<tbody>
<tr>
<td>( S_{LCb}^{1} = V_{1}^{LCb} I_{1}^{LCb} = 0 \text{ kVA} )</td>
<td>( S_{LCb} =</td>
</tr>
<tr>
<td>( P_{LCb}^{1} = V_{1}^{LCb} I_{1}^{LCb} \cos \phi_{1}^{LCb} = 0 \text{ kW} )</td>
<td>( P_{LCb} = \sum_{h=1}^{n} V_{h} I_{h} \cos \phi_{h} = 0 \text{ kW} )</td>
</tr>
<tr>
<td>( Q_{1}^{LCb} = V_{1}^{LCb} I_{1}^{LCb} \sin \phi_{1}^{LCb} = 0 \text{ kVAR} )</td>
<td>( D_{1}^{LCb} = V_{1}^{LCb} I_{1}^{LCb} = 3 \text{ kVA} )</td>
</tr>
<tr>
<td>( \phi_{1}^{LCb} = 0^\circ )</td>
<td>( D_{V}^{LCb} = V_{1}^{LCb} I_{1}^{LCb} = 0 \text{ kVA} )</td>
</tr>
<tr>
<td></td>
<td>( S_{LCb}^{H} = V_{H}^{LCb} I_{H}^{LCb} = 3 \text{ kVA} )</td>
</tr>
<tr>
<td></td>
<td>( N_{SB} = \sqrt{(S_{LCb}^{H})^2 - (P_{LCb})^2} = 3\sqrt{2} \text{ kVA} )</td>
</tr>
</tbody>
</table>
There is a disconnection in the IEEE’s power theory between the non-fundamental power terms at the load and the source; which agrees with the existing disconnections in Budeanu’s distortion power and Fryze’s reactive power.

Although the power theory embedded in the IEEE Standard 1459-210 has the support of the IEEE; the theory is internally inconsistent, is inconsistent with the PoCoE, the balance principle of the reactive power and [41] proves the falsehood of the following two statements in the IEEE Standard: “the non-active power N is not reactive power” and that “only when the waveforms are perfectly sinusoidal, N=Q₁=Q”. However, [41] proves that: even with non-sinusoidal waveforms the non-active power N can be reactive power.

5.3.5 Analysis of the Power Phenomena According to the CPC Power Theory

Equations (5.33) to (5.35) provide the power quantities of circuit 5.2SOFF according to the CPC power theory. In these Equations the voltage and current quantities are RMS values and they are estimated according the definitions given in Chapter Two, i.e. Equations (2.58) to (2.66). The super index letters \( LC – RLC \) and \( LCB – LC \) denote respectively, the equivalent admittance of the \( LC – RLC \) branches and the equivalent admittance of the \( LCB – LC \) branches. For example, \( Y^{LC-RLC}_1 \) denotes the equivalent admittance at fundamental frequency after combining the \( LC \) and \( RLC \) branches into one, thus,
\[
\begin{align*}
Y_{1}^{\text{CPC-RLC}} &= 0.1 + j0 & Y_{3}^{\text{CPC-RLC}} &= 0.9 - j0.3 \\
G_{1}^{\text{CPC-RLC}} &= 0.1 & G_{3}^{\text{CPC-RLC}} &= 0.9 & G_{e}^{\text{CPC-RLC}} &= \frac{P_{S}}{\|V_{S}\|^2} = 0.5 \\
\|I_{S}^{S}\| &= \frac{P_{S}}{\|V_{S}\|} = 50\sqrt{2}A & P_{S} &= 10kW \\
\|I_{S}^{S}\| &= 40\sqrt{2}A & D_{S}^{S} &= 8kVA \\
\|I_{S}^{S}\| &= 30A & Q_{r}^{S} &= 3\sqrt{2}kVAr & S^{S} &= \sqrt{182}
\end{align*}
\]

\[
\begin{align*}
Y_{1}^{\text{RLC}} &= 0.1 + j0.3 & Y_{3}^{\text{RLC}} &= 0.9 - j0.3 \\
G_{1}^{\text{RLC}} &= 0.1 & G_{3}^{\text{RLC}} &= 0.9 & G_{e}^{\text{RLC}} &= 0.5 \\
\|I_{a}^{\text{RLC}}\| &= \frac{P_{RLC}}{\|V_{S}\|} = 50\sqrt{2}A & P_{RLC} &= 10kW \\
\|I_{S}^{\text{RLC}}\| &= 40\sqrt{2}A & D_{S}^{\text{RLC}} &= 8kVA \\
\|I_{T}^{\text{RLC}}\| &= 30\sqrt{2}A & Q_{r}^{\text{RLC}} &= 6kVAr & S^{\text{RLC}} &= 10\sqrt{2}
\end{align*}
\]

\[
\begin{align*}
Y_{1}^{\text{LC}} &= -j0.3 & Y_{3}^{\text{LC}} &= 0 \\
G_{1}^{\text{LC}} &= 0 & G_{3}^{\text{LC}} &= 0 & G_{e}^{\text{RLC}} &= 0 \\
\|I_{a}^{\text{LC}}\| &= \frac{P_{LC}}{\|V_{S}\|} = 0A & P_{LC} &= 0kW \\
\|I_{S}^{\text{LC}}\| &= 0A & D_{S}^{\text{LC}} &= 0kVA \\
\|I_{T}^{\text{LC}}\| &= 30A & Q_{r}^{\text{LC}} &= 3\sqrt{2}kVAr & S^{\text{LC}} &= 3\sqrt{2}kVA
\end{align*}
\]

Notice how different is the magnitude of the reactive power using the CPC’s definition \(Q_{r}^{S} = 3\sqrt{2}kVAr\), the IEEE’s definition \(Q_{1}^{S} \approx 0kVAr\), Fryze’s definition \(Q_{r}^{S} \approx 9kVAr\) and Budeanu’s definition \(Q_{r}^{S} = 3kVAr\). Notice also that in contrast to the IEEE Standard, the CPC power theory yields a nonzero value for the reactive power which is in-line with the power theories of Budeanu and Fryze. Interestingly, the \(LC-branch\)’s reactive power is identical to the result attained with Fryze’s power theory, and the non-active power \(N^{S}\) of the IEEE standard, i.e. \(Q_{r}^{LC} = Q_{F}^{LC} = N^{S} = 3\sqrt{2}kVAr\), see Equations
(5.25), (5.30) and (5.35); also, the $RLC$ branch’s scattered power $D_S^{RLC} = 8kVA$ is identical to Budeanu’s distortion power $D_B^{RLC} = 8kVA$ see Equation (5.20). In the next section, the power theory presented in this dissertation explains these similarities among many others.

Examination of Equation (5.35) confirms that the $LC$ branch’s reactive current is $\|I_r^{LC}\| = 30A$ and agrees with Fryze’s estimate of reactive power, i.e. $Q_r^{LC} = Q_f^{LC} = 3\sqrt{2}kVAR$ but disagrees with Budeanu’s estimate of reactive power.

The results attained with the CPC power theory leads to conclude that it is possible to improve the power factor by placing a compensator with a reactive power value of $Q_r = 3\sqrt{2}kVAR$. More importantly, the CPC power theory provides a method to determine the compensator [53]. The application of this method yields the $LCb$ branch in Figure 5.2 but as the method is cumbersome, a simpler method is used in the next section to find this $LCb$ branch ($C = \frac{9}{80}$, $F$ and $L = \frac{80}{9}H$). Examination of Equations (5.33) to (5.35) show that the CPC power theory does not satisfy the PoCoE as the reactive power at the source cannot be attained with the values of the reactive power at the $LC$ and $RLC$ branches, i.e. $Q_r^S = 3\sqrt{2}kVAR \neq Q_r^{LC} + Q_r^{RLC}$. Equations (5.34) to (5.37) show the power quantities of circuit 5.2SON when using the CPC’s power theory.

\[
\begin{align*}
Y_1^{LCb} &= 0 & Y_3^{LCb} &= j0.3 \\
G_1^{LCb} &= 0 & G_3^{LCb} &= 0 & G_{e}^{RLC} &= 0 \\
\|I_a^{LCb}\| &= \frac{P^{LC}}{\|V^S\|} = 0A & P^{LCb} &= 0kW \\
\|I_s^{LCb}\| &= 0A & D_s^{LCb} &= 0kVA \\
\|I_r^{LCb}\| &= 30A & Q_r^{LCb} &= 3\sqrt{2}kVAR \\
 & & S_r^{LCb} &= 3\sqrt{2}kVA \\
\end{align*}
\]
Note that indeed the \( \text{LC} - \text{branch} \), with reactive power value \( Q_{r}^{\text{LC}} = 3\sqrt{2} \text{kVAR} \) releases the source from having a current component that is out-of-phase with the voltage as Equation (5.37) shows, which causes a zero reactive power on the source, i.e. \( Q_{r}^{\text{SB}} = 0 \text{kVAR} \). Notice that the reactive power of the \( \text{LC} - \text{branch} \) is identical to the reactive power of the \( \text{LC} - \text{branch} \), i.e. \( Q_{r}^{\text{LC}} = Q_{r}^{\text{LC}} = 3\sqrt{2} \text{kVAR} \), which coincides with Fryze’s power theory; however, according to Equation (5.34) the \( \text{RLC} - \text{branch} \) only requires \( Q_{r}^{\text{RLC}} = 6 \text{kVAR} \). Therefore, if these two branches indeed compensate the \( \text{RLC} - \text{branch}'s \) reactive power then \( Q_{r}^{\text{LCb}} \) and \( Q_{r}^{\text{LC}} \) must add geometrically and not arithmetically but this implies that \( Q_{r}^{\text{LCb}} \) and \( Q_{r}^{\text{LC}} \) are of different nature which cannot be accepted as this implication is contrary to the CPC’s dentition of reactive power i.e. \( Q_{r} = \|v\| \|i_{r}\| \). As the definition involves the multiplication of two scalar numbers then \( Q_{r}^{\text{LCb}} \) and \( Q_{r}^{\text{LC}} \) are also scalar and therefore \( Q_{r}^{\text{LCb}} \) and \( Q_{r}^{\text{LC}} \) must be added arithmetically not geometrically. Interestingly, the current on each branch involves a different harmonic so neither branch can aid the current requirement of the other branch; thus, according to the PoCoE the total reactive power for these two \( \text{LC} \) branches is \( Q_{r} = Q_{r}^{\text{LCb}} + Q_{r}^{\text{LC}} = 6\sqrt{2} \text{kVAR} \). Unfortunately, when the two branches are combined into one, the CPC power theory concludes that the reactive power on the equivalent branch is \( Q_{r} = 6 \text{kVAR} \) instead of \( Q_{r} = 6\sqrt{2} \text{kVAR} \). This is a contradiction as the power quantities in a branch must remain unchanged unless the voltage or current change. The CPC power theory provides two reactive power values on a circuit branch under constant voltage and current conditions. Thus, the contradicting results in this example show that the CPC power theory is internally inconsistent as Equation (5.38) confirms.
Although Czarnecki’s CPC power theory is the most fruitful power theory today and the IEEE honoured his author with the distinction of Fellow; the theory is internally inconsistent, is inconsistent with the PoCoE, the balance principle of the reactive power and other flaws have been pointed in [42].

Although the example above shows that the PoCoE cannot be satisfied using Budeanu, Fryze, The IEEE Standard or Czarnecki’s CPC power theory; the actual problems lay on the supporting structure on which the present power theories are based, i.e. the limitations of the mathematical framework – the algebra of complex numbers. Notice that the definitions of non-active power quantities are not signed quantities; therefore, the principle of conservation of energy can never be corroborated. As a consequence, it is also impossible to corroborate the balance principle of the reactive power. This explains why, until today, none of the existing power theories is able to provide a decomposition of the traditional definition of apparent power that accounts for the total non-active power and satisfy conservation at the same time [54].

The present power theories are based on a deficient framework; in short, as discussed in Chapter Three, the mathematical tool is inappropriate for describing the phenomena of power in non-sinusoidal conditions. These deficiencies and their consequences are succinctly described below.

1. The transformation operation used to transfer time-domain signals to and from the frequency-domain embeds an ambiguity and does not codify the entire information of

\[
\begin{align*}
Y_1^{Lb} &= -j0.3 & Y_1^{LC} &= 0 & Y_1^{Lb-LC} &= -j0.3 \\
Y_3^{Lb} &= 0 & Y_3^{LC} &= j0.3 & Y_3^{Lb-LC} &= j0.3 \\
\|V_1^{Lb}\| &= 100\sqrt{2}V & \|V_1^{LC}\| &= 100\sqrt{2}V & \|V_1^{Lb-LC}\| &= 100\sqrt{2}V \\
I_1^{Lb} &= -30A & I_1^{LC} &= 0A & I_1^{Lb-LC} &= -30A \\
I_3^{Lb} &= 0A & I_3^{LC} &= 30A & I_3^{Lb-LC} &= 30A \\
\|I_1^{Lb}\| &= 30A & \|I_1^{LC}\| &= 30A & \|I_1^{Lb-LC}\| &= 30\sqrt{2}A \\
Q_r^{Lb} &= 3\sqrt{2}kVAR & Q_r^{LC} &= 3\sqrt{2}kVAR & Q_r^{Lb-LC} &= 6kVA
\end{align*}
\]
the time-domain signal. As a consequence of this, in non-sinusoidal situations, it is not possible to use the superposition principle and it is not possible to corroborate Kirchhoff’s circuit’s laws and the PoCoE.

2. The present non-sinusoidal circuit analysis framework yields different groups of results (sets) that lack laws of operations, i.e. these sets are mathematically unrelated. This leads to a number of concepts in the frequency-domain that require a more rigorous mathematical treatment and to postulate power theories that are not consistent with the PoCoE.

5.3.6 Circuit Analysis in the $G_N$ Domain and Analysis of the Power Phenomena According to the Power Theory Proposed in this Dissertation

Application of Equation (4.29) to the excitation source given by Equation (5.16) yields,
\[
\begin{align*}
\text{Time Domain} & \\
\nu_{s1}(t) &= 100\sqrt{2}\sin \omega t \\
\nu_{s3}(t) &= 100\sqrt{2}\sin(3\omega t) \\
v^S(t) &= \nu_{s1}(t) + \nu_{s3}(t) \\
v^S(t) &= 100\sqrt{2}(\sin \omega t + \sin 3\omega t)
\end{align*}
\]
\[
\begin{align*}
\text{$G_N$ domain} & \\
\nu_{s1} &= -100\sigma_2 \\
\nu_{s3} &= 100 \sum_{i=1,i\neq 2}^{3+1} \sigma_i \\
v^S &= \nu_{s1} + \nu_{s3} \\
v^S &= 100(-\sigma_2 + \sum_{i=1,i\neq 2}^{3+1} \sigma_i)
\end{align*}
\] (5.39)

In contrast to the frequency-domain, the signal $\nu(t) = 100\sqrt{2}(\sin \omega t + \sin 3\omega t)$ does have a representation in the $G_N$ domain. Figure 5.5 shows the analysis of circuit 5.2SON where switches $S_1, S_2$ and $S_3$ are all closed. The analysis for each circuit resulting from operating switches $S_1, S_2, S_3$ is not required as the technique allows viewing what happens when the switches are operated. Note that opening switch $S_1$ and keeping $S_2$ and $S_3$ closed yields circuit 5.2SOFF and consequently the circulation of the third order harmonic current $-30\sigma_2\sigma_3\sigma_4$ is forced through the voltage source as Figure 5.6 shows. Opening switch $S_2$ while keeping $S_1$ and $S_3$ closed, forces the circulation of the fundamental current $30\sigma_1$ through the voltage source; opening switch $S_3$ while
keeping closed $S_1$ and $S_2$ releases the source from the current terms $-10\sigma_2$ and $90\sigma_1\sigma_3\sigma_4$ but forces the circulation of the fundamental current $30\sigma_1$ and the third harmonic current $-30\sigma_2\sigma_3\sigma_4$ through the voltage source.

\[ e(t) = 100\sqrt{2}[\sin(\omega t) + \sin(3\omega t)] \quad \longleftrightarrow \quad E = 100[-\sigma_2 + \sigma_1\sigma_3\sigma_4] \]

**Figure 5.5** Analysis in the $\mathcal{G}_V$ domain of the circuits in Figure 5.2. One diagram can describe the flow of all the harmonic currents resulting from the seven combinations of switches $S_1, S_2$ and $S_3$.

\[ e(t) = 100\sqrt{2}[\sin(\omega t) + \sin(3\omega t)] \quad \longleftrightarrow \quad E = 100[-\sigma_2 + \sigma_1\sigma_3\sigma_4] \]

**Figure 5.6** Analysis in the $\mathcal{G}_V$ domain of the circuit 5.2SOFF. The diagram shows how the current term $-30\sigma_2\sigma_3\sigma_4$ of the $LCb$-branch reroutes through the voltage source to satisfy Kirchhoff’s current law.

Equations (5.40) and (5.41) provide the circuit and power quantities for each branch of the circuit in Figure 5.2. Equations (5.40) and (5.41) allow examining the power
phenomena of all the seven circuits, including circuits 5.2SOFF and 5.2SON, resulting from the seven combination of switches \( S_1, S_2 \) and \( S_3 \); an unthinkable feature in the frequency-domain.

\[
\begin{align*}
V^S &= 100(-\sigma_2 + \Lambda_{i=1,2}^{3+1} \sigma_i) \\
I^{RLC} &= 30\sigma_1 - 10\sigma_2 + 90\sigma_1\sigma_3\sigma_4 - 30\sigma_2\sigma_3\sigma_4 \\
I^{LC} &= -30\sigma_1 \\
I^{LCb} &= 30\sigma_2\sigma_3\sigma_4 \\
I^{LCb-LC} &= -30\sigma_1 + 30\sigma_2\sigma_3\sigma_4 \\
I_g &= -10\sigma_2 + 90\sigma_1\sigma_3\sigma_4 \\
I^S &= -10\sigma_2 + 90\sigma_1\sigma_3\sigma_4 - 30\sigma_2\sigma_3\sigma_4 \\
I^{SB} &= -10\sigma_2 + 90\sigma_1\sigma_3\sigma_4 \\
\|V^S\| &= 100\sqrt{2} \\
\|I^{RLC}\| &= 100 \\
\|I^{LC}\| &= 30 \\
\|I^{LCb}\| &= 30 \\
\|I^{LCb-LC}\| &= 30\sqrt{2} \\
\|I_g\| &= 10\sqrt{82} \\
\|I^S\| &= 10\sqrt{91} \\
\|I^{SB}\| &= 10\sqrt{82}A \\
\end{align*}
\]

\[
\begin{align*}
M^{RLC} &= [10 + 6\sigma_1\sigma_2 + 6\sigma_3\sigma_4 + 8\Lambda_{i=1}^{4} \sigma_i]kVA \\
M^{LC} &= [-3\sigma_1\sigma_2 - 3\sigma_3\sigma_4]kVA \\
M^{LCb} &= [-3\sigma_1\sigma_2 - 3\sigma_3\sigma_4]kVA \\
M^{LCb-LC} &= -6\sigma_1\sigma_2 - 6\sigma_3\sigma_4 \\
M_g &= \langle M_g \rangle_0 + CN_d = [10 + 8\Lambda_{i=1}^{4} \sigma_i]kVA \\
M^S &= [10 + 3\sigma_2\sigma_4 + 3\sigma_3\sigma_4 + 8\Lambda_{i=1}^{4} \sigma_i]kVA \\
M^{SB} &= [10 + 8\Lambda_{i=1}^{4} \sigma_i]kVA \\
\|M^{RLC}\| &= \sqrt{236}kVA \\
\|M^{LC}\| &= 3\sqrt{2}kVA \\
\|M^{LCb}\| &= 3\sqrt{2}kVA \\
\|M^{LCb-LC}\| &= 6\sqrt{2}kVA \\
\|M_g\| &= \sqrt{164}kVA \\
\|M^S\| &= \sqrt{182}kVA \\
\|M^{RLC}\| &= \sqrt{164}kVA \\
\end{align*}
\]

The power quantities of circuit 5.2SOFF according to the \( G_N \) domain power theory, involve \( M^{RLC}, M^{LC} \text{ and } M^S \) in Equation (5.41); the reactive power multivector is \( \langle M^S \rangle_2 = 3\sigma_1\sigma_2 + 3\sigma_3\sigma_4 \) and the magnitude of the reactive power multivector is \( \|\langle M^S \rangle_2\| = \|CN_r\| = 3\sqrt{2}kVAR \); which is in line with the CPC power theory. This demonstrates that circuit 5.2SOFF does not have zero reactive power as estimated by the IEEE Standard 1459-2010. More importantly yet, notice that the reactive power has two components that add geometrically, \( CN_{r(ps)} = 3\sigma_1\sigma_2 \) and \( CN_{r(hi)} = 3\sigma_3\sigma_4 \) as inferred, yet not substantiated, by the CPC power theory. The reactive power term, \( CN_{r(ps)} = 3\sigma_1\sigma_2 \) results from the phase-shift between the voltage’s third harmonic term \( V_3 = 100\sigma_1\sigma_3\sigma_4 \) and a portion of the current’s third harmonic term \( I_3 = 30\sigma_2\sigma_3\sigma_4 \). This power term agrees with Budeanu’s power theory, it can be attained in the frequency-domain and is
seen on Equation (5.17) as $S_{3}^{S} = V_{3}^{S} I_{3}^{S} = (9 + j3)kVA$, where $Q_{3}^{S} = Q_{B}^{S} = j3kVAr$ (Equation 5.19). The term $CN_{r(hi)} = 3\sigma_{2}\sigma_{4}$ results from the interaction between the voltage’s fundamental component $V_{1} = -100\sigma_{2}$ and a portion of the current’s third harmonic $I_{3} = -30\sigma_{2}\sigma_{3}\sigma_{4}$. This term is one of the major contributions of this dissertation and it cannot be found in the frequency-domain using Steinmetz’s framework [41, 42]. The reason is straightforward; in non-sinusoidal situations, operations among quantities of different frequency are not defined in the frequency-domain. The term $CN_{d} = 8\Lambda_{\theta_{1}}^{4}\sigma_{i}$ in the $M^{RLC}$ and $M^{S}$ multivectors results from the addition of two products. The first product composed by the voltage and current terms $V_{3} = -100\sigma_{2}$ and $I_{1} = 90\sigma_{1}\sigma_{3}\sigma_{4}$ yields $CN_{d} = 9\Lambda_{\theta_{1}}^{4}\sigma_{i}$ while the second product composed by the voltage and current terms $V_{3} = 100\sigma_{1}\sigma_{3}\sigma_{4}$ and $I_{1} = -10\sigma_{2}$ yields $CN_{d} = -1\Lambda_{\theta_{1}}^{4}\sigma_{i}$. Notice that in the frequency-domain these two products involve quantities of different frequencies; thus, this result cannot be attained in the frequency-domain either. These results show clearly the imperative necessity of having a well-defined set of laws of operation among the quantities that result from each harmonic analysis. The magnitude of the $LC$-branch’s reactive power is $\|M^{LC}\| = 3\sqrt{2}kVA$ which is in line with the CPC and Fryze’s power theories. The circuit’s reactive current is $-30\sigma_{2}\sigma_{3}\sigma_{4}$ which agrees with the analysis in the frequency-domain and proves that Fryze’s power theory miscalculates the circuit’s reactive current, described in Equation (5.23) as $\|i_{r}^{S}\| = 10\sqrt{41}A \approx 64.03A$. The operation $I^{RLC} + I^{LC}$ yields $I^{S}$ and shows consistency with the Kirchhoff’s current laws while the operation $M^{RLC} + M^{LC}$ which yields $M^{S}$ shows consistency with the PoCoE.

The power quantities of circuit 5.2SON according to the $g_{N}$ domain power theory involve $M^{RLC}, M^{LC}, M^{Lb}$ and $M^{Sb}$ in Equation (5.41). It can be seen that the $LCb$-branch entirely compensates the circuit’s reactive power; which is in line with the CPC and Budeanu’s power theories and the analysis in the frequency-domain. This demonstrates that the reactive power of both, the $LCb$-branch and the $LC$-branch has two components that add geometrically, $CN_{r(ps)} = 3\sigma_{1}\sigma_{2}$ and $CN_{r(hi)} = 3\sigma_{3}\sigma_{4}$ as inferred, yet not
substantiated, by the CPC and Fryze’s power theories. The term $CN_d = 8 \sum_{i=1}^{4} \sigma_i$ in the $M^{RLC}$ and $M^{Sb}$ multivectors appears as the distortion power $D^{RLC}_B = 8kVA$ in Budeanu’s power theory, as reactive power $Q^{Sb}_F = 8kVAR$ in Fryze’s power theory, as the non-active power $N^{Sb} = 8kVA$ in the IEEE Standard 1459-2010 and as the scattered power $D^{RLC}_S = 8kVA$ in the CPC power theory. The operation $I^{LCb} + I^{LC} + I^{RLC}$ yields $I^{Sb}$ and shows consistency with the Kirchhoff’s current laws while the operation $M^{LC} + M^{LCb} + M^{RLC}$ which yields $M^{Sb}$ shows consistency with the PoCoE.

The power quantities of the circuit resulting from opening switch $S_3$ while keeping closed $S_1$ and $S_2$ involve $M^{LC}$ and $M^{LCb}$ in Equation (5.41). In this circuit the source’s power multivector is given by $M^{S(LC-LCb)} = M^{LC} + M^{LCb}$ and its magnitude yields $\|M^{S(LC-LCb)}\| = \|M^{LCb-LC}\| = 6\sqrt{2}kVA$, which ratifies the inconsistency of the CPC power theory discussed previously. Notice that the current on each branch involves a different harmonic so neither branch can aid the current requirement of the other branch; thus, according to the PoCoE the total reactive power for these two $LC$ branches is $Q_r = Q^{LCb}_r + Q^{LC}_r = 6\sqrt{2}kVAR$ as provided by the $G_N$ domain power theory. Applying Equations (3.12) and (3.16) which resulted from the PoCoE and could not be computed in the frequency-domain due to its lack of laws of operation also yield $6\sqrt{2}kVA$ as the reactive power magnitude of the circuit composed by branches $LCb$ and $LC$.

The results attained with the $G_N$ domain power theory have the following outstanding implications.

1. There are other forms of reactive power generation beside the one attained by the phase-shift between voltage and current of the same frequency [41, 42]. In other words, certain harmonic interactions yield another form of reactive power. This form of reactive power is reported in the scientific literature for the first time in [41].

2. Not every interaction among harmonics produces reactive power but only certain interactions. Notice that while certain interactions among harmonics yield $CN_d$ other interactions yield $CN_{r(hi)}$.
3. Steinmetz’s circuit analysis framework provides insufficient information to estimate the actual value of the reactive power. This discovery is astonishing as the frequency-domain has never been challenge before.

4. The lack of laws of operation in the frequency-domain in non-sinusoidal situations leads to postulate power theories where it is impossible to perform an accurate account of power quantities.

Notice how different is the magnitude of the reactive power for the circuit 5.2SOFF depending on which power theory is used:

\[
\begin{align*}
Q_B &= 3kVAR \\
Q_F &= \sqrt{82}kVAR \equiv 9kVAR \\
Q_1 &= 0kVAR \\
Q_r &= 3\sqrt{2}kVAR \\
\|CN_{r(ps)} + CN_{r(hi)}\| &= 3\sqrt{2}kVAR
\end{align*}
\]

\[
\begin{align*}
\text{Power theory author} \\
Budeanu &\text{ (5.42a)} \\
Fryze &\text{ (5.42b)} \\
(Team)\text{IEEE Standard 1459 – 2010} &\text{ (5.42c)} \\
Czarnecki &\text{ (5.42d)} \\
Castro – Núñez &\text{ (5.42e)}
\end{align*}
\]

The obvious question is: which one of these theories provides the correct value? Judging from Kuhn’s five criteria and the PoCoE’s perspective the answer is straightforward as only the $G_N$ domain power theory is consistent with the PoCoE. Additionally, according to Kuhn’s five criteria, the $G_N$ domain power theory is accurate, simple, internally and externally consistent with the PoCoE, Kirchhoff’s circuit laws, the principle of superposition and the balance principle of the reactive power and it is of broad scope and fruitful as demonstrated below and in [40], [41] and [42].


Power measurement is essential for the assessment of: energy consumption, efficient use of energy, and power quality, among many others. The measurement of power as well as many other electrical quantities is performed by an electrical instrument. The primary
function of any electrical instrument is to measure and indicate the value of the electrical quantity measured [55]. In sinusoidal situations, the rate of energy transfer can be measured by an analog or a digital instrument and the measurement of active average power, i.e. \( P = V I \cos \phi \), is often required. The electrodynamometer-type mechanism – an analog device and the basis of the watt-hour meter – can be used to measure volts, amperes and watts, which are real quantities in the electrical system as they can be defined for each instant of time [55]. Other quantities, e.g. reactive power, are a mathematical convenience implemented in the measuring instrument [55]. In non-sinusoidal situations the measurement of active average power performed with an electrodynamometer-type mechanism is wrong [56]. The reactive power measured by an electrodynamometer-type mechanism is attained by shifting the voltage signal 90°, whether in sinusoidal or non-sinusoidal conditions [55]. The arrival of the microprocessor brought the possibility of implementing a variety of methods – which were not previously possible – for measuring electrical quantities; however, under different conditions, e.g. non-sinusoidal situations, these different methods produce different results [57]. The measurement of power quantities in non-sinusoidal conditions is no longer a problem of digital sampling, or insufficient computational speed and capacity [58]. Indeed, sampling theory is currently embedded in almost any electric energy meter [59]. However, the computation of the reactive power quantity is still based on the 90° phase-shift method [59, 60] whether an analog or a digital meter is used. If measuring the rms values of voltage and current in non-sinusoidal conditions is not a problem and if there is also no problem in measuring the active average power in non-sinusoidal conditions, the obvious question is: where is the problem in the measurement of power quantities in non-sinusoidal conditions? The problem is simple to enunciate but very serious and difficult to solve; some of the power quantities in non-sinusoidal conditions have more than one definition [58], e.g. the definition of reactive power – a situation thoroughly explained in Chapters Two and Five of this dissertation. Moreover, to make things worse, the computational algorithms used in some standard measuring equipment developed to compute certain power quantities, e.g. the reactive power quantity, do not adhere to any of the proposed definitions [58, 61]. The problem reduces to achieving a proper definition of all the power quantities in non-
sinusoidal conditions [61]. This problem is well recognized in the literature and it has been approached from many different angles as evidenced by the samples of approaches given in [56] to [67]. All power theories are essentially a mathematical model aiming at interpreting, through an equation, the physical phenomena of power [8]. However, as mathematical models, power theories must aim at describing – through mathematics – different aspects of the power phenomena [68]. Therefore, the acceptance of a power theory and the assessment of its practical value should be based on the power theory’s consistency with well-established principles and laws of physics as the measurement of power quantities in non-sinusoidal conditions is a yet-to-be-resolved problem. The circuit in Figure 5.7 – also examined in [42] – provides evidence of the $G_N$ domain power theory’s practical application and its consistency with the principle of superposition, Kirchhoff’s circuit laws, the balance principle of the reactive power and the principle of conservation of energy in non-sinusoidal conditions. The voltage source in Figure 5.7 is given by,

$$v_s(t) = 100\sqrt{2}[\sin(\omega t) + 4\sin(5\omega t) + 2\sin(7\omega t) + \sin(11\omega t)]$$  \hspace{1cm} (5.44)  

![Figure 5.7](image_url)

**Figure 5.7** Circuit with a highly distorted voltage excitation source and highly distorted current

Applying Equation (4.29) to Equation (5.44) yields,
\[
\begin{align*}
\{ V_s &= \left[ -100\sigma_2 + 4 \sum_{i=1}^{6} \Lambda_{i=2}^{i=2} \sigma_i + 2 \sum_{i=1}^{8} \Lambda_{i=2}^{i=2} \sigma_i + 1 \sum_{i=1}^{12} \Lambda_{i=2}^{i=2} \sigma_i \right] V \tag{5.45a} \\
\|V\| &= 100.105V \tag{5.45b}
\end{align*}
\]

The impedance at each harmonic is given by,
\[
\begin{align*}
Z_1 &= 1.02 - 2.1\sigma_1\sigma_2 \tag{546a} \\
Z_5 &= 1.02 - 10.5\sigma_1\sigma_2 \tag{546b} \\
Z_7 &= 1.02 - 14.7\sigma_1\sigma_2 \tag{546c} \\
Z_{11} &= 1.02 - 23.1\sigma_1\sigma_2 \tag{546d}
\end{align*}
\]

The current through the voltage source is given by,
\[
\begin{align*}
I_1 &= -18.714\sigma_2 - 38.529\sigma_1 \tag{5.47a} \\
I_5 &= 37E^{-3} \sum_{i=2}^{6} \Lambda_{i=2}^{i=2} \sigma_i - 38E^{-2} \sum_{i=2}^{6} \Lambda_{i=2}^{i=2} \sigma_i \tag{5.47b} \\
I_7 &= 94E^{-4} \sum_{i=2}^{8} \Lambda_{i=2}^{i=2} \sigma_i - 13E^{-2} \sum_{i=2}^{8} \Lambda_{i=2}^{i=2} \sigma_i \tag{5.47c} \\
I_{11} &= 19E^{-4} \sum_{i=2}^{12} \Lambda_{i=2}^{i=2} \sigma_i - 43E^{-3} \sum_{i=2}^{12} \Lambda_{i=2}^{i=2} \sigma_i \tag{5.47d} \\
I_g &= -18.714\sigma_2 + 37E^{-3} \sum_{i=2}^{6} \Lambda_{i=2}^{i=2} \sigma_i + 94E^{-4} \sum_{i=2}^{8} \Lambda_{i=2}^{i=2} \sigma_i + 19E^{-4} \sum_{i=2}^{12} \Lambda_{i=2}^{i=2} \sigma_i \tag{5.47e} \\
I_b &= -38.529\sigma_1 - 38E^{-2} \sum_{i=2}^{6} \Lambda_{i=2}^{i=2} \sigma_i - 13E^{-2} \sum_{i=2}^{8} \Lambda_{i=2}^{i=2} \sigma_i - 43E^{-3} \sum_{i=2}^{12} \Lambda_{i=2}^{i=2} \sigma_i \tag{5.47f} \\
I_s &= I_1 + I_g + I_b = I_1 + I_5 + I_7 + I_{11} \tag{5.47g} \\
\|I_s\| &= 42.84A \tag{5.47h}
\end{align*}
\]

The power multivector at the source is given by \(M^s = V_sI_s\), where \(V_s\) is given by Equation (5.45a) and \(I_s\) is given by Equation (5.47g); the result yields,
In Equations (5.48a) to (5.48i) the super index denotes source. Notice that the source’s active average power is $P^s = 1871.548\,W$ and this result is identical to the one attained with Equation (2.5) which defines the active average power in the time domain. Since it is impossible for an electrodynamometer-type watt meter to measure the correct power in non-sinusoidal conditions [56] then the reading in a practical analog meter is different from the value $P^s = 1871.548$; consequently, both, the active average power and the energy measured by a practical analog watt-hour meter is erroneous. However, as a digital watt-hour meter computes the power at the source by following Equation (2.5), then the resultant active average power results from $P^s = P_1^s + P_5^s + P_7^s + P_{11}^s = 1871.548\,W$ which is in line with Equation (5.48a).

The partial results of the reactive power $Q$ in the frequency domain are: $Q_1 = j3852.92$, $Q_5 = j1.51$, $Q_7 = j0.27$ and $Q_{11} = j0.04$; these results provide only the reactive power due to phase-shift. As there are no rules of operation for these frequency-domain elements, then, from the rigorous mathematical perspective, no operation can be performed among them [40]. In an electrodynamometer-type watt meter and in many digital watt meters the reactive power is found with the following Equation [61],

$$Q = \frac{1}{\tau_o} \int_{-\tau_o/2}^{+\tau_o/2} i(t) v \left( t - \frac{\pi}{2} \right) dt = \sum_{h=1}^{n} V_h l_h \cos \left( \varphi_n - h \frac{\pi}{2} \right)$$

(5.49)

If the watt meter is used to measure power at the source then $i(t)$ is the current.
through the source, while \( v(t) \) is the source’s voltage. Notice the 90° shift in the voltage signal \( v(t) \) in Equation (5.49) and the similarity with Equation (2.5) to calculate the value of the active average power \( \langle p(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} i(t)v(t)dt = \sum_{h=1}^{n} V_h I_h \cos(\varphi_h) = \sum_{h=1}^{n} P_h \). Notice that Equation (5.49) does not satisfy any of the reactive power definitions given previously in Chapters Two and Five. In general, Equation (5.49) does not satisfy any definition of reactive power [61]. Thus, a practical analog or digital watt meter will read a reactive power value of \( Q = -3854.619 \text{VAR} \) but many other digital watt meters will provide a different value depending on definition of reactive power embedded in the microprocessor of the multifunctional digital device. Consequently, the energy consumption for which a customer is billed today depends partially on the definition of reactive power embedded in the microprocessor of the multifunctional digital device and not on the actual energy consumption.

According to the \( G_N \) domain power theory, the total reactive power due only to the phase-shift is \( CN^s_{r(ps)} = [-3852.92 - 1.51 + 0.27 + 0.04] \sigma_1 \sigma_2 = -3854.159 \sigma_1 \sigma_2 \), where \( CN^s_{r(ps)} \) results from grouping all the \( \sigma_1 \sigma_2 \) terms resulting from the product of \( V_s \) given by Equation (5.45a) and \( I_b \) given by Equation (5.47f), thus \( V_s I_b = CN_{r(ps)} + CN_{r(hi)} \). Notice that a number of other terms different from \( \sigma_1 \sigma_2 \) result in the calculation of the reactive power; these terms are immersed in the multivector \( CN_{r(hi)} \) given by Equation (5.48d). Nevertheless, how can we assess whether the \( G_N \) domain power theory provides the right results? One approximation, – already considered in Chapter Five – is to prove that the \( G_N \) domain power theory fulfills well-established principles, such as the Principle of Conservation of Energy and adheres to well-established laws, such as Kirchhoff’s Circuits laws and the balance principle of the reactive power. Adherence to well-establish principles and laws – on behalf of the model under examination – is a necessary condition because as a surrogate of the physical phenomenon, a model must mimic the behavior of the physical phenomenon under study [69]. For this reason, normally, the first step in developing a mathematical model is to identify which physical property of the physical phenomenon is being conserved or balanced [70] – recall
objective three on the research’s objective in Section 1.3. In electric systems, power – in all its forms, i.e. active, reactive, distorted, etc. – is the physical property that needs to be kept balanced. When all the power quantities are kept balanced at any instant of time, compliance with the PoCoE is assured since conservation laws are special cases of balanced laws [70]. Balance laws are mathematical expressed as [70],

$$\frac{dQ(t)}{dt} = q_{in}(t) + g(t) - q_{out}(t) - c(t).$$ (5.50)

In Equation (5.50) $Q(t)$ depicts the physical property being monitored, $q_{in}(t)$ the influx of $Q(t)$, $q_{out}(t)$ the efflux of $Q(t)$, $g(t)$ the rate at which $Q$ is generated, $c(t)$ the rate at which $Q$ is consumed and $t$ depicts the independent variable. Therefore, in cases where the generation and consumption within the system boundaries is zero, i.e. $g = c = 0$, then the balance law described by Equation (5.50) becomes a conservation law [70].

$$\frac{dQ(t)}{dt} = q_{in}(t) - q_{out}(t)$$ (5.51)

Notice also in Equation (5.50) that if both, the influx and the efflux are zero, i.e. $q_{in}(t) = q_{out}(t) = 0$ and consequently $\frac{dQ(t)}{dt} = 0$, then the rate at which $Q$ is generated must equal the rate at which $Q$ is consumed, thus,

$$g(t) = c(t)$$ (5.52)

Equation (5.52) requires that for the principle of conservation of energy to be fulfilled in electric systems; the volt-amperes generated at the source must be equal to the volt-amperes consumed at the load. This is equivalent to saying that at any instant of time each one of the power terms at the source must equal its correspondent power term at the load, thus,
The power multivector at the source impedance is given by $M^{sz} = V_{sz} I_s$, where $I_s$ is given by Equation (5.47g) and $V_{sz}$ is the voltage across the source impedance. The term $V_{sz}$ can be calculated as $V_{sz} = \sum_{h=1}^{n} Z_{szh} I_{sh}$ where $Z_{szh} = 0.02 - 0.1 \omega_h \sigma_1 \sigma_2$ and $I_{sh}$ denotes the individual harmonic components of $I_s$. Thus,

$$V_{sz} = V_{sz1} + V_{sz5} + V_{sz7} + V_{sz11}$$

(5.54a)

$$V_{sz1} = 1.101 \sigma_1 - 4.227 \sigma_2$$

(5.54b)

$$V_{sz5} = 0.189 \Lambda_{i=2}^{6} \sigma_i + 0.011 \Lambda_{i=2}^{6} \sigma_i$$

(5.54c)

$$V_{sz7} = 0.095 \Lambda_{i=2}^{6} \sigma_i + 0.004 \Lambda_{i=2}^{6} \sigma_i$$

(5.54d)

$$V_{sz11} = 0.048 \Lambda_{i=2}^{6} \sigma_i + 0.001 \Lambda_{i=2}^{6} \sigma_i$$

(5.54e)

$$\| V_{sz} \| = 4.37V$$

(5.54f)

The power multivector at the load is given by $M^l = V_l I_s$, where and $I_s$ is given by Equation (5.47g) and $V_l$ is the voltage across the load and can be calculated via

$$M^{sz} = V_{sz} I_s = P^{sz} + CN^{sz}_{r(ps)} + CN^{sz}_{r(hi)} + CN^{sz}_d$$

(5.55a)

$$P^{sz} = 36.697$$

(5.55b)

$$CN^{sz}_{r(ps)} = -183.529 \sigma_1 \sigma_2$$

(5.55c)

$$CN^{sz}_{r(hi)} = -5.865 \Lambda_{i=3}^{8} \sigma_i - 3.149 \Lambda_{i=3}^{8} \sigma_i - 1.648 \Lambda_{i=3}^{12} \sigma_i + A^{sz}$$

(5.55d)

$$A^{sz} = -0.062 \sigma_1 \sigma_2 \Lambda_{i=7}^{8} \sigma_i - 0.026 \sigma_1 \sigma_2 \Lambda_{i=7}^{12} \sigma_i$$

(5.55e)

$$\| CN^{sz}_{r(hi)} \| = 6.86$$

(5.55f)

$$CN^{sz}_d = -3.389 \Lambda_{i=1}^{6} \sigma_i - 1.737 \Lambda_{i=1}^{8} \sigma_i - 0.930 \Lambda_{i=1}^{12} \sigma_i + B^{sz}$$

(5.55g)

$$B^{sz} = 0.00 \Lambda_{i=1}^{12} \sigma_i + 0.005 \Lambda_{i=9}^{8} \sigma_i + 0.00 \Lambda_{i=9}^{12} \sigma_i + 0.004 \Lambda_{i=1}^{12} \sigma_i$$

(5.55h)

$$\| CN^{sz}_d \| = 3.92$$

(5.55i)
Kirchhoff’s current law as $V_1 = V_3 - V_{sz}$. Thus, 

$$
\begin{align*}
V_1 &= V_{i1} + V_{i5} + V_{i7} + V_{i11} \\
V_{i1} &= -1.101\sigma_1 - 95.773\sigma_2 \\
V_{i5} &= 3.811 \sum_{i=2}^{6} \sigma_i - 0.011 \sum_{i=2}^{6} \sigma_i \\
V_{i7} &= 1.905 \sum_{i=2}^{8} \sigma_i - 0.004 \sum_{i=2}^{6} \sigma_i \\
V_{i11} &= 0.952 \sum_{i=2}^{12} \sigma_i - 0.001 \sum_{i=2}^{6} \sigma_i \\
\|V_1\| &= 95.88V
\end{align*}
$$

\( (5.56a) \)
\( (5.56b) \)
\( (5.56c) \)
\( (5.56d) \)
\( (5.56e) \)
\( (5.56f) \)

$$
\begin{align*}
M^l &= V_{sz}I_s = V_{sz}I_l = P^l + CN^l_{r(ps)} + CN^l_{r(hi)} + CN^l_d \\
P^l &= 1834.851 \\
CN^l_{r(ps)} &= -3670.587\sigma_1\sigma_2 \\
CN^l_{r(hi)} &= -110.512 \sum_{i=3}^{6} \sigma_i - 60.37 \sum_{i=3}^{8} \sigma_i - 32.5629 \sum_{i=3}^{12} \sigma_i + A^l \\
A^l &= -1.235\sigma_1\sigma_2 \sum_{i=7}^{8} \sigma_i - 0.524\sigma_1\sigma_2 \sum_{i=7}^{12} \sigma_i \\
\|CN^l_{r(hi)}\| &= 130.07 \\
CN^l_d &= -67.798 \sum_{i=1}^{6} \sigma_i - 34.751 \sum_{i=1}^{8} \sigma_i - 17.595 \sum_{i=1}^{12} \sigma_i + B^l \\
B^l &= 0.008 \sum_{i=7}^{12} \sigma_i + 0.106 \sum_{i=7}^{12} \sigma_i - 0.004 \sum_{i=7}^{12} \sigma_i + 0.082 \sum_{i=7}^{12} \sigma_i \\
\|CN^l_d\| &= 78.19
\end{align*}
$$

\( (5.57a) \)
\( (5.57b) \)
\( (5.57c) \)
\( (5.57d) \)
\( (5.57e) \)
\( (5.57f) \)
\( (5.57g) \)
\( (5.57h) \)
\( (5.57i) \)

Notice that the addition of each power term in Equation (5.57) and Equation (5.55) yields the correspondent power term in Equation (5.48); consequently the balance principle of power is fulfilled and as a result the principle of conservation of energy is also fulfilled. At present, no power theory is capable of fulfilling Equation (5.52) and therefore, no power theory is presently able to comply with the principle of conservation of energy or the balance principle of the reactive power. The two questions that arise immediately are:

1) What is the practical value of a power theory – or mathematical model – that does not mimic the behavior of the physical phenomena under study and consequently it does not adhere to well-established physical principles?

2) What are the scientific community’s parameters to assess the practical value of a mathematical model – power theory – intended to explain the physical phenomena of
power in non-sinusoidal conditions?

5.4.1 The Practical Problem of Reactive Power Compensation and its Importance in Electric Networks

Although today, in sinusoidal situations, the power factor and the reactive power are well-accepted definition by the scientific community; it took, after 1888, more than forty years and the work of Steinmetz, Houston, Kennely, Iliovici, Budeanu, Emde, Knowlton and Fortescue, among many others, for these two concepts – the power factor and the reactive power – to be accepted [71]. Today, the practical problem of reactive power compensation in sinusoidal situations is a simple task taught throughout the world commonly in a second-year course of electrical engineering programs; however, the practical problem of reactive power compensation in non-sinusoidal situations is a yet-to-resolve problem [72]. Utility Engineers install shunt capacitors to reduce power losses, regulate power bus voltage and improve power quality, e.g. power factor improvement [73]; however, placing shunt capacitors in today’s network can negatively impact the electric network, i.e. increase harmonic distortion, reduce power factor, and create resonance among many others problems [74]. The maintenance of the voltage profile is amongst the most important problems in energy generation and transmission [75]; therefore, the importance of this problem obliges for strategies capable of supplying the non-sinusoidal electric network with the proper amount of reactive power. The problem of reactive power supply whether in harmonic or sinusoidal conditions is usually addressed as an optimization problem [76-79]; unfortunately, the majority of these techniques assume sinusoidal conditions [73] but in fact, today, the supply waveform in the electric network is normally non-sinusoidal [80]. More recently, microgrids [81] and FACTS devices [75] have been proposed for reactive power compensation and harmonic compensation, although traditionally, active filters [82] are normally used for harmonic suppression. Although active filters do improve power quality, they also inject non-sinusoidal currents [82]; therefore, reactive power compensation by passive elements is still preferred because it does not inject distortions [53]. In summary, it is well recognized
in the technical literature that reactive power measurement is essential for voltage control and power factor improvement among other aspects [61-81]; however, how to measure a quantity for which there is currently no agreement on its definition? Proper reactive compensation can be achieved once an accurate definition of reactive power is recognized [72]. The question by Makram, Haines and Girgis [61, 72, 74] is significant: “how to define power components in non-sinusoidal situation?” According to Makram, Haines and Girgis, “once the reactive power is accurately calculated, a procedure for designing the shunt capacitor in the presence of harmonics and distortion can be established.” The example below provides sufficient evidence that indeed once the reactive power is accurately estimated; it is possible to determine the appropriate compensator. Notice that the $G_N$ domain power theory does not pre-establishes the necessity of a capacitor for reactive power compensation as in non-sinusoidal situations it is a combination of inductors and capacitors what determines the appropriate passive compensation device as the example below shows. It is clear that a minimum requisite for a proper definition for the reactive power quantity must involve compliance with the balance principle of the reactive power; however, none of the existing definitions comply with this well-established physical principle.

With the $G_N$ domain power theory the procedure for designing a compensator results from solving a set of $N$ equations and $n$ unknowns. In the case of the circuit of Figure 5.7, the set of two equations and two unknowns result from assuming a series $LC$ branch for the compensator and is given by,

\[
\begin{align*}
-95.773 \left( \frac{\omega C}{\omega^2 LC - 1} \right) \sigma_1 &= 38.529 \sigma_1 \quad (5.58a) \\
-3.811 \left( \frac{3\omega C}{9\omega^2 LC - 1} \right) \Lambda_{i=2}^6 \sigma_i &= -0.4 \Lambda_{i=2}^6 \sigma_i \quad (5.58b)
\end{align*}
\]

Solving Equation (5.58) yields $L = 1.874H$ and $C = 0.228F$; which is the compensator of Figure 5.8. Since the addition of the compensator changes the impedance viewed by the voltage source, then the current through the voltage source – now denoted $I_{scp}$ – changes to,
Figure 5.8  Circuit with a highly distorted voltage excitation source and highly distorted current, yet a simple LC branch improves the system’s power quality by reducing the current distortion through the voltage source and by improving the power factor from 0.4 to 0.998

Comparing Equation (5.47f) with Equation (5.59g) shows that the compensator severely diminishes the reactive current $I_b$ flowing through the voltage source. Also, the compensator increases the system’s efficiency as evidenced by the terms $-18.714\sigma_2$ in Equation (5.47a) and the term $-19.9053\sigma_2$ in Equation (5.59b). This increment in efficiency can be seen also by the increment in the source’s active average power denoted by $P^{scp}$ in the power multivector $M^{scp}$ calculated by the geometric product of the new
source current $I_{scp}$ given by Equation (5.59a) and the source’s voltage $V_s$ given by Equation (5.45a), thus

\[
\begin{align*}
M^{scp} &= V_s I_{scp} = P^{scp} + CN^{scp}_{r(ps)} + CN^{scp}_{r(hl)} + CN^{scp}_d \\
p^{scp} &= 1990.657 \\
CN^{scp}_{r(ps)} &= -60.024\sigma_1\sigma_2 \\
CN^{scp}_{r(hl)} &= 75.899 \lambda_{i=3}^6 \sigma_i + 26.177 \lambda_{i=3}^8 \sigma_i + 8.011 \lambda_{i=3}^{12} \sigma_i + A^{scp} \\
A^{scp} &= -2.657\sigma_1\sigma_2 \lambda_{i=7}^8 \sigma_i - 1.125\sigma_1\sigma_2 \lambda_{i=7}^{12} \sigma_i \\
\|CN^{scp}_r(hl)\| &= 80.74 \\
CN^{scp}_d &= -76.092 \lambda_{i=1}^6 \sigma_i - 38.906 \lambda_{i=1}^8 \sigma_i - 19.722 \lambda_{i=1}^{12} \sigma_i + \|B^{scp}\| \\
B^{scp} &= 0.007 \lambda_{i=7}^{12} \sigma_i + 0.107 \lambda_{i=7}^8 \sigma_i - 0.004 \lambda_{i=9}^{12} \sigma_i + 0.172 \lambda_{i=1}^{12} \sigma_i \\
\|CN^{scp}_d\| &= 87.71
\end{align*}
\]

Notice that the active average power has increased from $P^s = 1871.548W$, see Equation (5.48b), to $P^{scp} = 1990.657W$ – Equation (5.60b). Since the addition of the compensator changed the current through the voltage source from $I_s$ to $I_{scp}$, then the voltage across both, the source impedance and load impedance also changed. The voltage across the source impedance now denoted as $V_{zscp}$ can be found with the expression $V_{zscp} = \sum_{h=1}^{n} Z_{szh} I_{scp h}$ where $Z_{szh} = 0.02 - 0.1\omega_h\sigma_1\sigma_2$ and $I_{scp h}$ denotes the individual harmonic components of $I_{scp}$, in Equation (5.59a). The voltage across the load now denoted as $V_{lp}$ can be determined using Kirchhoff’s voltage law, i.e. $V_{lp} = V_s - V_{zscp}$. Thus,

\[
\begin{align*}
V_{zscp} &= V_{zscp1} + V_{zsck5} + V_{zsck7} + V_{zsck11} \\
V_{zsck1} &= 1.979\sigma_1 - 0.4555\sigma_2 \\
V_{zsck5} &= 0.3917 \lambda_{i=2}^5 \sigma_i + 0.002 \lambda_{i=2}^6 \sigma_i \\
V_{zsck7} &= 0.1915 \lambda_{i=2}^8 \sigma_i + 0.0009 \lambda_{i=2}^6 \sigma_i \\
V_{zsck11} &= 0.0945 \lambda_{i=2}^{12} \sigma_i + 0.0003 \lambda_{i=2}^6 \sigma_i \\
\|V_{zsck}\| &= 2.08V
\end{align*}
\]
The current through the compensator, denoted as $I_{cp}$, in the circuit of Figure 5.8 can be found with the expression $I_{cp} = \sum B_{cph} V_{lcph}$, where $B_{cph}$ is the compensator’s admittance at each harmonic and $V_{lcph}$ is the load voltage at each harmonic. Thus

\[
\begin{align*}
I_{cp} &= I_b = I_{cp1} + I_{cp5} + I_{cp7} + I_{cp11} \\
I_{cp1} &= -0.78813 \sigma_2 - 39.6392 \sigma_1 \\
I_{cp5} &= 2E^{-4} \Lambda_{i=1}^6 \sigma_i - 4247E^{-4} \Lambda_{i=2}^6 \sigma_i \\
I_{cp7} &= 1E^{-4} \Lambda_{i=1}^8 \sigma_i - 14473E^{-4} \Lambda_{i=2}^8 \sigma_i \\
I_{cp11} &= 0 \Lambda_{i=1}^{12} \sigma_i - 448E^{-4} \Lambda_{i=2}^{12} \sigma_i \\
\|I_{scp}\| &= 39.65A
\end{align*}
\] (5.63a)

The power multivector for the load, the compensator and the source impedance – denoted as $M^{lcp}$, $M^{cp}$, and $M^{szcp}$ respectively – are,

\[
\begin{align*}
M^{lcp} &= V_{lcpl} l = p^{lcp} + CN_{r(ps)}^{lcp} + CN_{r(hi)}^{lcp} + CN_d^{lcp} \\
p^{lcp} &= 1982.715 \\
CN_{r(ps)}^{lcp} &= -3966.262 \sigma_1 \sigma_2 \\
CN_{r(hi)}^{lcp} &= -109.57 \Lambda_{i=3}^6 \sigma_i - 59.953 \Lambda_{i=3}^8 \sigma_i - 32.329 \Lambda_{i=3}^{12} \sigma_i + A^{lcp} \\
A^{lcp} &= -1.11 \sigma_1 \sigma_2 \Lambda_{i=7}^{12} \sigma_i - 0.472 \sigma_1 \sigma_2 \Lambda_{i=7}^{12} \sigma_i \\
\|CN_{r(hi)}^{lcp}\| &= 129.01 \\
CN_d^{lcp} &= -64.818 \Lambda_{i=1}^6 \sigma_i - 33.447 \Lambda_{i=1}^8 \sigma_i - 17.045 \Lambda_{i=1}^{12} \sigma_i + B^{lcp} \\
B^{lcp} &= 0.007 \Lambda_{i=7}^{12} \sigma_i + 0.097 \Lambda_{i=7}^8 \sigma_i - 0.003 \Lambda_{i=9}^{12} \sigma_i + 0.074 \Lambda_{i=1}^{12} \sigma_i \\
\|CN_d^{lcp}\| &= 74.90
\end{align*}
\] (5.64a)
\[
M^{cp} = V_{lcpl}I_{lcpl} = P^{cp} + CN_{r(ps)}^{cp} + CN_{r(hl)}^{cp} + CN_{d}^{cp}
\]
\[
P^{cp} = 0
\] (5.65a)
\[
CN_{r(ps)}^{cp} = 3946.148\sigma_1\sigma_2
\] (5.65b)
\[
CN_{r(hl)}^{cp} = 185.308\Lambda_{i=3}^{6}\sigma_i + 86.096\Lambda_{i=3}^{8}\sigma_i + 40.352\Lambda_{i=3}^{12}\sigma_i + A^{cp}
\] (5.65c)
\[
A^{cp} = -1.29\sigma_1\sigma_2\Lambda_{i=7}^{8}\sigma_i - 0.546\sigma_1\sigma_2\Lambda_{i=7}^{12}\sigma_i
\] (5.65d)
\[
\|CN_{r(hl)}^{cp}\| = 208.28
\] (5.65e)
\[
CN_{d}^{cp} = -1.947\Lambda_{i=1}^{6}\sigma_i - 1.111\Lambda_{i=1}^{8}\sigma_i - 0.626\Lambda_{i=1}^{12}\sigma_i + B^{cp}
\] (5.65f)
\[
B^{cp} = 0.00\Lambda_{i=7}^{12}\sigma_i + 0.001\Lambda_{i=7}^{8}\sigma_i - 0.00\Lambda_{i=9}^{12}\sigma_i + 0.081\Lambda_{i=1}^{12}\sigma_i
\] (5.65g)
\[
\|CN_{d}^{cp}\| = 2.329
\] (5.65h)

\[
M^{s2cp} = V_{s2cp}I_{s2cp} = P^{s2cp} + CN_{r(ps)}^{s2cp} + CN_{r(hl)}^{s2cp} + CN_{d}^{s2cp}
\]
\[
P^{s2cp} = 7.942
\] (5.66a)
\[
CN_{r(ps)}^{s2cp} = -39.909\sigma_1\sigma_2
\] (5.66b)
\[
CN_{r(hl)}^{s2cp} = 0.161\Lambda_{i=3}^{6}\sigma_i + 0.015\Lambda_{i=3}^{8}\sigma_i - 0.012\Lambda_{i=3}^{12}\sigma_i + A^{s2cp}
\] (5.66c)
\[
A^{s2cp} = -0.257\sigma_1\sigma_2\Lambda_{i=7}^{8}\sigma_i - 0.108\sigma_1\sigma_2\Lambda_{i=7}^{12}\sigma_i
\] (5.66d)
\[
\|CN_{r(hl)}^{s2cp}\| = 0.32
\] (5.66e)
\[
CN_{d}^{s2cp} = -9.327\Lambda_{i=1}^{6}\sigma_i - 4.347\Lambda_{i=1}^{8}\sigma_i - 2.05\Lambda_{i=1}^{12}\sigma_i + B^{s2cp}
\] (5.66f)
\[
B^{s2cp} = 0.00\Lambda_{i=7}^{12}\sigma_i + 0.010\Lambda_{i=7}^{8}\sigma_i + 0.00\Lambda_{i=9}^{12}\sigma_i + 0.016\Lambda_{i=1}^{12}\sigma_i
\] (5.66g)
\[
\|CN_{d}^{s2cp}\| = 10.49
\] (5.66h)

Notice that a term-by-term addition of Equations (5.64), (5.65) and (5.66) yields also Equation (5.60); thus, the rate at which the volt-ampere is generated equals the rate at which the volt-ampere is consumed. Consequently, it is once again proved that the \(G_N\) domain power theory fulfills well-established principles, such as the Principle of Conservation of Energy and adheres to well-established laws, such as Kirchhoff’s Circuits laws and the balance principle of the reactive power. But more importantly, the line and the source impedance have no negative impact on the \(G_N\) domain power theory as it has been the case with other power theories, e.g. [83]. Additionally, the \(G_N\) domain power provides the means to improve the system’s power quality, another well-recognized problem in non-sinusoidal situations [74]. Equation (5.65) shows how the compensator releases the voltage source from almost all the reactive power required by
the load. Thus the magnitude of the power multivector is reduced from $\|M^s\| = 4287.51\,V\,\!A$ to $\|M^{scp}\| = 1995.13\,V\,\!A$. Consequently the system’s power factor is improved from $pf = \frac{1871.548}{4287.51} = 0.436$ to $pf = \frac{1990.657}{1995.13} = 0.998$.

5.4.2 The Inconsistency in the Traditional Definition of Apparent Power Related to Physical Power Flow Phenomena

The circuit analysis in the $G_N$ domain and its associated power theory deepens our understanding of the power phenomena in non-sinusoidal circuits. For instance, the circuit in Figure 4.8 is used by Czarnecki in [45] to present a thus far unresolved question. The circuit, and its analysis in the $G_N$ domain, is given in Figure 5.7. The analysis in the frequency-domain reveals that the power of the 1Ω-resistor, i.e. the source’s resistor, is $2000W$ of which $1600W$ are provided by the harmonic current source $J$ and $400W$ are provided by the voltage source $V$. On the other hand, the power of the 4Ω-resistor, is also $2000W$ of which $1600W$ are provided by the voltage source $V$ and $400W$ by the harmonic current source $J$. Consequently, the net flow of energy/power at the cross section $xx$ is $0W = 1600W - 1600W$. Additionally, since this is a resistive circuit, the reactive power measured at the cross section $xx$ is also zero; however, the value of the non-sinusoidal apparent power at the cross section $xx$ is $S = 4000VA \neq 0$ although $P = Q = 0$. Czarnecki’s question is certainly a challenge; in “terms of what power can this difference be explained” [45]. Until today, no power theory is capable of providing a satisfactory answer.
Figure 5.9 Analysis in the $G_N$ domain showing the flow of energy/power in the circuit. Although the active average power $P$ and the reactive power $Q$ at the cross section $xx$ are both zero, the traditional definition of apparent power gives $\|S^{xx}\| = 4000VA$. In contrast, the multivector power is $M^{xx} = 2400\sigma_1\sigma_2\sigma_3$ confirming that $P = Q = 0$ at the cross section $xx$. However, overruling the application of the correction factor $f$ when calculating the multivector power yields $M^{xx} = 3200 + 2400\sigma_1\sigma_2\sigma_3$ and $\|M^{xx}\| = 4000VA$. Thus the traditional definition of apparent power involves the gross energy/power exchange of active average power which in contrast to the frequency domain analysis in sinusoidal conditions.

A detailed explanation of the power phenomena in this circuit is provided in [40] using the $G_N$ domain power theory. The detailed calculation of the power at the cross section $xx$ is repeated here in Equation (5.67) to show the broadness and fruitfulness of the $G_N$ domain power theory.

$$v(t) = \sqrt{2}[100\sin(\omega t)] \rightarrow \langle V \rangle_1 = -100\sigma_2$$  
$$j(t) = 50\sqrt{2}\sin(2\omega t) \rightarrow \langle J \rangle_2 = 50\sigma_1\sigma_3$$  
$$I_s^b = -20\sigma_2 + 40\sigma_1\sigma_3$$  
$$V_{4\Omega} = -80\sigma_2 - 40\sigma_1\sigma_3$$  
$$I_{4\Omega} = -20\sigma_2 - 10\sigma_1\sigma_3$$

$$M^{xx} = -[80\sigma_2 + 40\sigma_1\sigma_3][-20\sigma_2 + 40\sigma_1\sigma_3] = 2400\sigma_1\sigma_2\sigma_3 \quad (5.67a)$$

where

$$[-80\sigma_2][-20\sigma_2] = 1600 \quad (5.67b)$$

$$[-80\sigma_2][40\sigma_1\sigma_3] = 3200\sigma_1\sigma_2\sigma_3 \quad (5.67c)$$

$$[-40\sigma_1\sigma_3][-20\sigma_2] = -800\sigma_1\sigma_2\sigma_3 \quad (5.67d)$$

$$[-40\sigma_1\sigma_3][40\sigma_1\sigma_3] = -1600 \quad (5.67e)$$
Equation (5.67) shows that the exchange of active average power from load to source is zero as pointed by Czarnecki [45] and the analysis in the frequency-domain. However, notice that if the application of the correction factor \( f \), i.e. Equation (4.68), is not applied to the product \([-40\sigma_1\sigma_2][40\sigma_1\sigma_2]\) in Equation (5.67e), then the result yields 1600\(W\) instead of \(-1600W\) as \((\sigma_1\sigma_2)^2 = -1\). In this case the power multi-vector becomes \([M^{xx} = 3200 + 2400\sigma_1\sigma_2\sigma_3]VA\) and therefore the magnitude is not \(|M^{xx}| = 2400\) but \(|M^{xx}| = 4000VA\), the value provided by the traditional definition of apparent power, i.e. \(|S^{xx}| = |V^{xx}| |I^{xx}| = 4000VA\). Consequently, in this particular example the traditional definition of apparent power measures the gross energy exchange at the cross section \(xx\) as demonstrated in [40]. It is evident from a frequency domain analysis, that the net flow of active average power at the cross section \(xx\) is 0\(W\), yet the traditional definition of apparent power contradicts the physical phenomena by measuring a nonzero value of active average power at the cross section \(xx\). This result proves further that the traditional definition of apparent power given by \(|S| = |V||I|\) needs to be revised, a point inferred, yet not demonstrated by Filipsky [22]. The reason why the power multivector is not zero, i.e. \(M^{xx} = P + CN_{r(ps)} + CN_{r(ht)} + CN_d \neq 0\), although \(P = CN_{r(ps)} = CN_{r(ht)} = 0\), is because the degrading power of the multivector power is not zero, i.e. \(CN_d = 2400\sigma_1\sigma_2\sigma_3\).

### 5.4.3 Undermining an Established Concept

There is yet another area in which the \(G_N\) domain power theory outshines the power theories available today and in particular the ones examined previously. It is currently accepted by the scientific community that the complexity of the compensator is related to the number of harmonics [16, 53]; however, [41] and [42] show, with two different examples, that the number of harmonics is not necessary linked to the complexity of the compensator. According to Emanuel [84], the circuit in Figures 5.8a with voltage source \(e_a(t) = 100\sqrt{2}\left[\sin(\omega t) + \frac{1}{11}\sin(11\omega t) + \frac{1}{13}\sin(13\omega t)\right]\) requires 15 reactive elements.
to achieve a near unity power factor while the circuit in Figures 5.8b with voltage source
\( e_b(t) = 100\sqrt{2}[\sin(\omega t) + 4 \sin(11\omega t) + 2 \sin(7\omega t) + \sin(11\omega t)] \) requires 28 reactive
elements. However, [41] and [42] show that in both cases a simple LC series branch is
sufficient to bring the power factor to a value above 0.99. A detailed analysis is provided
in [41] and [42] and it is omitted here to keep the thesis document within the required
space limits.

![Circuit Diagrams]

\[v_a(t) = 100\sqrt{2}[\sin(\omega t) + 4 \sin(11\omega t) + 2 \sin(7\omega t) + \sin(11\omega t)]\]

\[v_b(t) = 100\sqrt{2}\left[\sin(\omega t) + \frac{1}{11}\sin(11\omega t) + \frac{1}{13}\sin(13\omega t)\right]\]

Figure 5.10  Circuits showing that the number of harmonics in the excitation
source is not necessarily correlated to the number of reactive elements
required to bring the power factor to a near unity value.

5.5 Chapter Summary

A generalization of the circuit analysis in the \( G_N \) domain shows that \( G_N \) domain
power is consistent with the PoCoE. Although a comparison between Steinmetz’s circuit
analysis framework and the new technique proposed here show that both techniques give
the same magnitude for the current and voltage at each harmonic; the proposed technique
is externally consistent with Kirchhoff’s circuit laws and the principle of superposition
while Steinmetz’s circuit analysis framework is not. A comparison among \( G_N \) domain
power and four well-established power theories i.e. Budeanu, Fryze, IEEE standard and the CPC show that: while $G_N$ domain power is consistent with the PoCoE and the balance principle of the reactive power, all the other four power theories are not. In addition, it is also shown that the $G_N$ domain power has a better mathematical formality for the formulation of the power equation than the other four power theories.
Chapter Six: Conclusions and Future Work

6.1 Conclusions

The research work presented in this dissertation makes the following contributions to the solution of developing a power theory in non-sinusoidal conditions:

1. The mathematical weaknesses of the present circuit analysis framework are identified and the impact of these weaknesses on the present power theories is examined.
2. A new circuit analysis approach is developed to overcome the weaknesses encountered in Steinmetz’s framework.
3. Although the magnitude of the current and the voltage evaluated at each harmonic using the proposed framework are identical with established frameworks, the proposed circuit analysis approach permits detailed evaluation of the flow of currents and powers in a given circuit diagram; which has not been possible to date.
4. The new circuit analysis approach reveals that the Steinmetz’s and related frameworks do not provide sufficient information to perform an energy analysis.
5. The proposed approach provides a new, conservative power theory that is suitable for non-sinusoidal conditions and reduces to the well-known power equation in sinusoidal conditions.
6. The proposed approach identifies four reasons why the present definition of apparent power is unsuitable for developing a power theory in non-sinusoidal conditions.
7. The proposed approach provides a more efficient and versatile method to design compensators in non-sinusoidal conditions that is similar to the present method of compensation design in the sinusoidal case.

8. It is shown that the power multi-vector provides deep insights regarding current and voltage quantities, in contrast with the four power theories under examination. Also, errors in these four theories are identified.

9. It is shown that the power multi-vector may be interpreted to provide valuable information about apparent power that is still consistent with the traditional definition. The power multi-vector also can be used to identify errors in the traditional definition of apparent power.

10. The proposed approach provides a power equation that has an improved mathematical formality to calculate the power in sinusoidal and non-sinusoidal conditions compared to present frameworks.

### 6.2 Future Work

To date, there is no accepted solution on how to interpret the power phenomena in the following three conditions that bare future investigation:

- Three-phase circuit operation in sinusoidal conditions with unbalanced loads. Three theories exist today [85] to calculate the apparent power in this case; and, although the three theories provide the same figure under balanced conditions, they differ in unbalanced conditions.

- Three-phase circuits in non-sinusoidal conditions with balanced loads.

- Three-phase circuits in non-sinusoidal conditions with unbalanced loads.

The work presented in this dissertation can be further investigated to see if the proposed power theory can be expanded to three-phase circuits.
References


