A Study of the Multiple-Unicast Network Coding Conjecture Using Riemannian Manifolds

by

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Abstract

Network coding encourages information mixing at the intermediate nodes within a network. The multiple-unicast conjecture proposed by Li and Li [18] in 2004 is one of the most well-known unsolved problems in network coding field. The conjecture asserts that, for multiple independent unicast transmissions in an undirected network, network coding has no advantage over traditional routing. In this thesis, we study the conjecture by embedding graphs into Riemannian manifolds using a geometric framework developed by Xiahou et al. [32]. We prove that isometric embedding of graphs into a Riemannian manifold is impossible. Then, interestingly, we construct an embedding that achieves an infinitesimally small distortion. We show that if the multiple-unicast network coding conjecture is true on Riemannian manifolds, it is also true for undirected networks. Our hope is to develop a Riemannian geometry approach for making new progresses against the long-time open conjecture.
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<td>The multiple-unicast network coding conjecture</td>
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<tr>
<td>LP</td>
<td>Linear programming</td>
</tr>
<tr>
<td>$n$</td>
<td>Number of nodes</td>
</tr>
<tr>
<td>$k$</td>
<td>Number of unicast sessions</td>
</tr>
<tr>
<td>$G$</td>
<td>A graph $G$ or a network $G$</td>
</tr>
<tr>
<td>$\omega(e)$</td>
<td>A cost in link $e$</td>
</tr>
<tr>
<td>$E$</td>
<td>A set of edges</td>
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<tr>
<td>$e$</td>
<td>An edge in a graph $G$</td>
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<tr>
<td>$V$</td>
<td>A set of vertices</td>
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<tr>
<td>$v$</td>
<td>A vertex in a graph $G$</td>
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<tr>
<td>$s$</td>
<td>Source node in a single-source multicast network</td>
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<tr>
<td>$s_i$</td>
<td>Source node of $i^{th}$ pair of unicast</td>
</tr>
<tr>
<td>$t_i$</td>
<td>Receiver node of $i^{th}$ pair of unicast</td>
</tr>
<tr>
<td>$uv$</td>
<td>An undirected link in a network $G$</td>
</tr>
<tr>
<td>$\vec{uv}$</td>
<td>A directed link in a network $G$</td>
</tr>
<tr>
<td>$f$</td>
<td>A flow with a non-negative value in each edge</td>
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<tr>
<td>$r_i$</td>
<td>Desired throughput at node $i$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>An n-D Euclidean space</td>
</tr>
<tr>
<td>$C^\infty$</td>
<td>Infinitely differentiable</td>
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<td>$M$</td>
<td>A Riemannian manifold $M$</td>
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<td>$T_pM$</td>
<td>Tangent space of a Riemannian manifold $M$ at point $p$</td>
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<tr>
<td>$\gamma$</td>
<td>A smooth curve in a Riemannian manifold $M$</td>
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$g_p$  A Riemannian metric at point $p$

ODE  Ordinary differential equations

d  Length of shortest path between two nodes in a graph $G$

c  A shortest path in a Riemannian manifold $M$

l  Length of a shortest path in a Riemannian manifold $M$

r  Radius of a circle

u/v/p/q  A node in a graph $G$

u'/v'/p'/q'  A point in $M$ which models a node in $G$

U  Upper bound of length of the shortest path

L  Lower bound of length of the shortest path

$\mathcal{X}$  A set of source messages

$\mathcal{A}$  A set of directed links

$H(\mathcal{A})$  Entropy of messages transmitted in a set of links $\mathcal{A}$
Chapter 1

Introduction

1.1 Background

1.1.1 Network Coding

The fundamental principal of the traditional routing scheme in a computer communication network is that data is transmitted by moving packets independently in a shared network, from one host to the other. Packets are the basic information transmission unit in a communication network. Data for transmission from the source is divided into packets, and each packet is forwarded individually in the same or different paths to the intended destination. On their way to the terminal, the nodes they bypass are called intermediate nodes. Packets can be replicated at an intermediate node, and then further forwarded along multiple downstream paths. Upon arrival at the destination, packets are reassembled to the original information. Such a routing scheme is referred to as the store-and-forward scheme.

In 2000, Ahlswede et al. presented the idea of network coding. Instead of simply replicating and forwarding packets, with network coding, nodes can combine several packets together using information coding, e.g., through the bit-wise exclusive-or operation over two packets, or through more general forms of linear coding. It was shown that network coding enhances the capacity of an information network over routing. Compared to classic data routing, network coding opens up new opportunities in improving throughput, transmission efficiency, robustness and security of data transmission. The network is more robust since successful assembling the data no longer dependent on the reception of a single packet. The information can not be decoded unless sufficient packets are received, which improves data transmission security in the network. Given the growing demand of Internet bandwidth and
known applications of network coding, studies on network coding have attracted substantial attention from both academia and industry in the past 16 years.

1.1.2 The Multiple-Unicast Network Coding Conjecture

In the multiple-unicast setting, a group of source nodes wish to transmit information to their respective terminal nodes. Each source corresponds to a single terminal, and the information for communication from the unicast sessions is independent from one another. More specifically, there are a set of source nodes \( \{s_i\} \subseteq V \) and a set of terminal nodes \( \{t_i\} \subseteq V \). Each \( s_i \) is required to communicate to a corresponding \( t_i \). If the network is directed, network coding can improve the throughput for multiple-unicast sessions [18]. The situation is largely different for the case of undirected networks, where no advantage of network coding has been observed. The multiple-unicast network coding conjecture by Li and Li [18] in 2004 states that for multiple unicast sessions in an undirected network, network coding is equivalent to routing. The conjecture is one of the most important and intensively studied open problems in the field of network coding. In the past years, efforts for finding counter examples to the conjecture have proven sterile. On the positive side, the multiple-unicast conjecture has been verified in a number of special scenarios, including, for example, the case of two unicast sessions, the case where all sessions share one or two common terminal nodes, the case where all terminal nodes lie in a common face of a planar network, the Okamura-Seymour network \((K_{3,2})\) [13] and Hu’s 3-commodity network [4]. As arguably the most important open problem in the field of network coding [13], the conjecture calls for further investigation.

1.2 Motivations and Research Objectives

Xiahou et al. [32] exploits a newly developed framework, space information flow and aim to make progress along a final proof to the multiple-unicast conjecture through a geometric
approach. Their study embed a general network graph into Euclidean space with $O(\log k)$ distortion (the distortion of an embedding measures the factor by which the embedding distorts distances), prove that network coding has no advantage over routing in Euclidean space, and conclude that the coding advantage (the ratio of network coding throughput over routing throughput) is upper-bounded by $O(\log k)$. They also develop a methodology for proving the conjecture in a certain group of networks. As a potential research, they propose to further improve the known special case results is by first embedding graphs into Riemannian manifolds with a lower distortion, and proceed to prove the multiple-unicast network coding conjecture in Riemannian manifolds.

The work in this thesis starts from the above idea of embedding graphs into Riemannian manifolds, which appears to be promising for the following reasons. First, using Riemannian manifolds as a host space for graph embedding can result in a much lower distortion as compared to using Euclidean spaces, as we show later in the thesis. Furthermore, Xiahou et al. confirm the correctness of the multiple-unicast network coding conjecture (its equivalent version translated from throughput domain into the cost domain, such that the conjecture claims network coding has no advantage over routing for saving transmission bandwidth/cost while sustaining a given unicast throughput vector among specified source and terminal pairs in a metric space) in Euclidean space. Riemannian manifolds locally resembles Euclidean space; they share some properties which could potentially be used to prove the conjecture in Riemannian manifolds.

In the thesis, we are interested in the following two problems:

- Can we find a method to embed general graphs into Riemannian manifolds isometrically, or with a distortion arbitrarily close to 1?

In the first half of thesis, we intend to construct an embedding of general network topologies of undirected graphs into a continuous and more structured space that has a low distortion. Ideally, we want the distortion of embedding
general graphs into Riemannian manifolds to be 1 (no distortion), which makes it an isometric embedding. The geometric framework of Xiahou et al. uses a fundamental theorem of metric embedding that for every \( n \)-point metric space there exists an embedding into Euclidean space with distortion \( O(\log k) \). Given two metric spaces \((X, d_x)\) and \((Y, d_y)\), an injective mapping \( f : X \rightarrow Y \) is called an embedding of \( X \) into \( Y \). In our case, \( X \) is a graph metric with a shortest path between nodes as \( d_x \), and \( Y \) is a Riemannian manifold space with a minimal distance between points as \( d_y \). The distortion of an embedding is 1 if it is isometric, i.e., the distances are preserved between every pair of nodes, and each \( d_x \) is equal to the corresponding \( d_y \). A basic question is whether an isometric embedding for general graphs into Riemannian manifolds exists. In Chapter 4, we prove that such isometric embedding does not exist. However, interestingly, we are able to construct an embedding of graphs into Riemannian manifolds with a distortion arbitrarily close to 1 (the distortion can be \( 1 + \epsilon \) for some constant \( \epsilon \) arbitrarily close to 0 that is independent of the graph to be embedded). We use \( G = (V, E) \) to represent an undirected network where \(|V| = n\). Each edge \( e \in E \) has a non-negative weight that can be interpreted as its distance or cost. An \( n \times n \) metric space \( C \) is hence induced by computing the length of the shortest path \( d_{uv} \) between each pair of nodes \( u, v \in V \). We use \( M \) to denote the host Riemannian manifold for graph embedding. We use \( \gamma_{uv} \) to represent the shortest path between two points \( u \) and \( v \) in the Riemannian manifold \( M \) and \( l_{\gamma_{uv}} \) for the length of \( \gamma_{uv} \). We will construct an embedding of an edge-weighted undirected graph \( G \) using a Riemannian manifold \( M \), satisfying

\[
l_{\gamma_{uv}} \leq d_{uv} \leq (1 + \epsilon)l_{\gamma_{uv}}, \text{ where } \epsilon \text{ is arbitrarily close to 0}
\]

which has a smaller distortion compared to the one from embedding graphs
into Euclidean space:

\[ \| u, v \| \leq d_{uv} \leq O(\log k) \| u, v \| \]

The distortion is reduced from \( O(\log k) \) to a constant arbitrarily close to 1. Equipped with such a distortion, we can then proceed to prove that if the multiple-unicast network coding conjecture is true in Riemannian manifolds, then it is also true in undirected networks.

- Can we show that network coding cannot outperform routing in some specific groups of Riemannian manifolds?

In the second half of this thesis, we apply the geometric framework of Xiahou et al. \cite{32} to prove that the coding advantage is 1 in some elementary models of Riemannian manifolds. By projecting the shortest paths in Euclidean space (Euclidean distance) from high dimension to 1-D, their study shows the cost domain in high dimension is the linear combination of the cost domain in 1-D space. Since the multiple-unicast conjecture is true in 1-D Euclidean space, they draw the conclusion that the multiple-unicast network coding conjecture is true in Euclidean space of dimension \( n \), for any \( n \geq 1 \). Inspired by their approach, we aim to prove that the multiple-unicast conjecture is true in some constant curvature 2-D Riemannian manifolds, where the constant curvature can be positive (sphere), zero (Euclidean space) or negative (hyperbolic space).

1.3 Contributions

The main contributions of this thesis are listed below.

- We construct an embedding of general graphs into Riemannian manifolds, with a distortion arbitrarily close to 1.
• We show that if the multiple-unicast network coding conjecture is true in Riemannian manifolds, then it is also true in undirected networks. This opens a new direction for studying and hopefully proving the multiple-unicast conjecture in general networks, from a perspective of Remannian geometry.

• We verify the multiple-unicast conjecture in some elementary Riemannian manifolds, including circles, 2-D spheres and cylindrical tubes.

1.4 Thesis Organizations

The rest of this thesis is organized as follows. Chapters 2 provides some theoretical foundations of network coding, graph embedding theory and Riemannian manifolds. Chapter 3 is an overview of the literature related to network coding and the multiple-unicast conjecture. We embed general graphs into Riemannian manifolds in Chapter 4. Chapter 5 presents the proofs of the conjecture in some special cases of Riemannian manifolds. The last chapter concludes this thesis and outlines future work.
Chapter 2

Preliminaries

2.1 Network Coding

A communication network has an underlying graph topology $G = (V, E)$. A vertex $v \in V$ represents a node in a network, and an edge $e \in E$ stands for a communication link. Each edge can be capacity-weighted, specifying the maximum rate where information can be transmitted across that link. Below we introduce the principle of network coding, and use an example to demonstrate the power of network coding. The term *coding advantage* refers the ratio of network coding throughput over routing throughput.

2.1.1 Network Coding Introduction and Examples

Communication networks today are mostly based on the principle of packet switching, where multiple packets travel over the same network and share (compete for) resources in the network [8]. Routing is base on this fundamental assumption. Instead of simply forwarding data, network coding is applied to make nodes able to recombine several input packets into several encoded output packets. A common form of network coding is linear network coding, in which packets are encoded through symbol-wise linear operations over a finite field.

While linear combination requires more computational power at the nodes of the network, modern computing nodes in the communication are getting more advanced, and computing resources can further be acquired as a pay-as-you-go service from Internet data centers that are becoming ubiquitous around different parts of the world. Complementary to node capacity enhancement in the network, network coding can help making more efficient use of the precious link resources, by exploiting computation power at network nodes.

Departing from simple store-and-forwarding, network coding encourages nodes in a net-
work to combine a number of packets into encoded packets for further transmission along downstream links. We describe how linear network coding works in a network in the following.

When packets arrive at an intermediate node, they can be mixed into one or more recombined packets. Based on the work of Fragouli et al. [8], let $W^1, W^2, \ldots, W^n$ be the original packets. In linear network coding, each packet is associated with a sequence of coefficients $j_1, j_2, \ldots, j_n$ in $\mathcal{F}_2$, i.e., $X = \sum_{i=1}^{n} j_i W^i$. The encoding can be applied recursively at multiple nodes. As enough packets arrive at a terminal node, the decoding procedure can initiate to recover the original information. To decode the packets with received set $(j_1, X^1), \ldots, (j_m, X^m)$, we need to solve the linear system \{\begin{align*} X^j &= \sum_{i=1}^{n} j_i W^i \end{align*}\}, where $W^i$ are the unknowns. In other words, we need to discover the original packets $W^1, W^2, \ldots, W^n$ by decoding the encoded packets $X^1, X^2, \ldots, X^m$ which form a linear system with $m$ equations and $n$ unknowns. It is feasible to recover the original data only if $m \geq n$.

According to Fragouli et al. [8], a simple encoding algorithm, known as randomized network coding, is to have each node selected its coefficients in its encoding vectors uniformly at random, in a complete independent and decentralized manner, amenable for practical application. We can also use a polynomial time deterministic algorithm, since each node uses fixed linear coefficients, and the packets only need to carry the information vectors. In practical applications, to limit the size of the linear system and hence the complexity of the Gaussian elimination procedure, original data at the source are often divided into generations, such that encoding and decoding are both confined to packets from the same generation. Real-world implementations of network coding often choose to perform encoding and decoding operations over a finite field $\mathcal{F}_{2^s}$, such that each symbol is $ns$ the a binary string of $s$ bits. Addition is simple bit-wise xor. Multiplication can be performed by referring to a multiplication table of size $2^s \times 2^s$.

In summary, network coding can fundamentally change the way a communication network
operates, enabling transmission efficiency and throughput gain. Other benefits of network coding, such as robustness, adaptability and security, are not further elaborated since this thesis focuses on the throughput gain of network coding.

We review an example of network coding to illustrate its benefit over traditional routing, in terms of throughput gain. The butterfly network as shown in Fig. 2.1 is a classic example of multicast network that demonstrates the basic idea behind network coding. A multicast network transmits information from a single source node to a specific group of terminal nodes. It is proved that the coding advantage is upper-bounded in undirected multicast networks by 2 by Li et al. [19]. Each edge has unit capacity in this 7-node 9-edge example network. Fig. 2.1 demonstrates the data transmission in the butterfly network with traditional routing. The color-filled nodes represent the source and receiver nodes. $s$ is the source node and there are two receiver nodes $t_1$ and $t_2$. The intermediate nodes $v_1, v_2, v_3, v_4$ in the white circles help to forward the packets to receivers with no interest of knowing the data. They can also help replicate and/or encode data whenever necessary. Assume the source node $s$ needs to send packets $x$ and $y$ to both $t_1$ and $t_2$. In Fig. 2.1, only routing is performed. If $s$ conveys information to $t_1$, $v_3$ forwards packet $y$ to $v_4$ as shown in Fig. 2.1 (a). When $s$ transmits data to $t_2$, $v_3$ sends packet $x$ to $v_4$. With routing, an intermediate node can only simply forward or replicate the packets before forwarding. Thus, $v_3$ can send either $x$ or $y$. $v_3$ has to send $x$ and $y$ independently to $v_4$. However, the situation changes when network coding is enabled. With network coding, intermediate nodes have the capability to combine packets and forward. As shown in Fig. 2.2, $v_3$ combines the packets it receives from $v_1$ and $v_2$ then sends $x + y$ to $v_4$. $t_1$ receives $x$ from $v_1$ and $x + y$ from $v_4$. $t_1$ can now calculate and recover $y$ from those information. Similarly for $t_2$, by receiving $y$ from $v_2$ and $x + y$ from $v_4$, it can recover both $x$ and $y$. $v_3$ is only required to send data one time with network coding. This example illustrates how network coding can enhance network capacity and save bandwidth.
Figure 2.1: The butterfly network with traditional routing.

Figure 2.2: The butterfly network with network coding.
2.1.2 Network Coding in Undirected Multiple-Unicast Networks

Since its proposal in 2000, network coding has spawned a large volume of studies on both theory and applications. Coding advantage in the network of unicast, broadcast and multicast was soon investigated \cite{2,24}. However, the case of network coding in undirected multiple-unicast networks remains unclear. Li and Li \cite{18} examined network coding in multiple-unicast sessions. They show that for directed multiple-unicast networks, the coding advantage is unbounded. However, for the case of undirected networks, the coding advantage is unknown. They propose the multiple-unicast network coding conjecture that states that for multiple-unicast sessions in an undirected network, network coding can not outperform conventional routing. In the throughput domain, the conjecture is that a throughput vector is feasible for multiple independent unicast sessions in an undirected network with network coding if and only if it is feasible with routing. It can be translated into the cost domain: in an undirected network \((G,w)\) with cost-weighted edges, we have \(f\) to be the flow vector of a network coding solution for \(k\) independent unicast sessions with throughput vector \(r\), then \(\sum_e (f(e)\omega(e)) \geq \sum_i (d_ir_i)\). A series of studies have been devoted to this open problem, which remains open till today, and has become known as an important and hard problem in the theory of network coding.

From 2004 to 2016, no counter examples were discovered to the multiple-unicast conjecture. At the same time, the conjecture was only verified for a number of special types of multiple-unicast networks. Xiahou \textit{et al.} \cite{32} study the multiple-unicast network coding conjecture from a geometry perspective. They create a framework for proving multiple-unicast network coding conjecture which can be applied to several types of networks. Their framework consists of four steps. In step one, they translate the conjecture from throughput domain to cost domain by LP duality. Next, the conjecture is translated from cost domain to space domain by performing graph embedding into high dimensional Euclidean space (or another geometric space). In the third step, they reduce the problem from a high dimension
space to 1-D space. Lastly, they construct a direct proof for 1-D scenario. They study the upper-bound for the distortion of the general case, where in the second step, for each node \( u \) in \( G \), \( u \) is mapped to \( d(u, A_i) \mid i = 1, ..., O(\log^2 k) \), where it applies the Corollary from Linial et al. [20] and shows that \( \frac{d(u,v)}{O(\log k)} \leq \|u, v\| \leq d(u, v) \). The result presents an \( O(\log k) \) distortion in the embedding from \( G \) to \( f_1^{O(\log^2 k)} \). Consequently, it proves that for \( k \) unicast sessions in an undirected capacitated network \((G, E)\) with \( n \) vertices, the coding advantage is upper-bounded by \( O(\log k) \). They further mention the potential approach to reduce the distortion by embedding graphs into Riemannian manifolds instead of Euclidean space.

As the last puzzle in completing coding advantage in general communicational networks, we think it is crucial to continue the study in every potential direction. Continuing with the proposal by Xiahou et al., in this thesis, we will study on the multiple-unicast network coding conjecture from the perspective of Riemannian manifold. First we give the model and notations for an undirected multiple-unicast network.

In an undirected multiple-unicast network \( G(V, E) \), each \( v \in V \) corresponds to a node in the network. In a multiple-unicast scenario, there are \( k \) pairs of sources \( s_i \) and terminals \( t_i \) where \( 1 \leq i \leq k \). These \( k \) pairs of unicast sessions are independent, each with a desire throughput \( r_i \). \( e \in E \) stands for each undirected link, with \( c \in \mathbb{Q}_{+}^{|E|} \) being the link capacity vector and \( \omega \in \mathbb{Q}_{+}^{|E|} \) being the link cost vector where \( \mathbb{Q} \) is a set of positive rational numbers.

A link \( uv \) consists of a pair of directed links \( \vec{uv} \) and \( \vec{vu} \). Let \( A \) denote the set of directed links in \( G \), where \( A = \{\vec{uv}, \vec{vu} \mid \forall(u, v) \in E\} \). The rate of information flow being transmitted in \( G \) is represented by a flow vector \( f \in \mathbb{Q}_{+}^{A} \). Thus \( f(e) \) stands for the flow rate of an undirected link \( e = (u, v) \) where \( f(e) \) is the combination of \( f(\vec{uv}) \) and \( f(\vec{vu}) \). We have

\[
f(e) = f(\vec{uv}) + f(\vec{vu})
\]
2.2 Graph Embedding

The first step for our thesis is to attempt to embed general graphs isometrically or at least arbitrarily isometrically to Riemannian manifold. A well-known theorem from Bourgain [5] shows an \( n \)-point finite metric space embeds into Euclidean space with an \( O(\log n) \) distortion. Bourgain’s result considers the worst case distortion and later Linial et al. [20] prove that embedding the metrics of constant-degree expander graphs into Euclidean space requires an \( O(\log n) \) distortion. As to the average case, Abraham et al. [1] strengthen Bourgain’s result by providing an embedding with constant average distortion into Euclidean space for arbitrary metric spaces.

2.3 Riemannian Manifold

Locally, Riemannian manifold resembles Euclidean space. Xiahou et al. concluded in their study that the multiple-unicast conjecture is true in Euclidean space. However, according to graph embedding theory, the distortion of graphs embedding into Euclidean space is \( O(\log n) \). It is not sufficient to prove the conjecture in general graphs. We believe a Riemannian manifold is a better host space for proving the conjecture in general case. The strategy is to first prove that if the conjecture is true in Riemannian manifolds then it is true in graphs. This is more promising than using Euclidean space since Euclidean space is a special case of Riemannian manifolds with constant zero curvature. In other words, 2-D Riemannian manifolds are a more general group of space. Embedding into Riemannian manifolds is more likely to achieve a smaller distortion. Our goal in the first step is to achieve a small enough constant distortion. The second step is to prove that the multiple-unicast conjecture is true in Riemannian manifolds.

Before introducing Riemannian manifolds, we first review the definitions of manifolds and smooth manifolds.
**Definition 1.** (Lee, 2010 [17]) An $n$-dimensional manifold is a topological space that locally resembles Euclidean space of dimension $n$, i.e., each point in the manifold has a neighborhood that is homeomorphic to the $n$-D Euclidean space.

**Definition 2.** (Mukherjee, 2015 [23]) A smooth manifold of dimension $n$ is a topological manifold which is locally isomorphic to the space $(\mathbb{R}^n, C^\infty)$.

A smooth manifold is a topological manifold but not necessarily vice versa. In this thesis, we use Riemannian manifolds that are a special class of smooth manifolds further equipped with a Riemannian metric, such that geometric quantities such as angle, distance and area can be defined. Here are the definitions of tangent space and Riemannian manifolds.

**Definition 3.** (Weisstein, 2002 [26]) Let $x$ be a point in an $n$-dimensional compact manifold $M$ and attach at $x$ a copy of $\mathbb{R}^n$ tangential to $M$. The resulting structure is called the tangent space of $M$ at $x$ and is denoted by $T_pM$. If $\gamma$ is a smooth curve passing through $x$, then the derivative of $\gamma$ at $x$ is a vector in $T_pM$.

**Definition 4.** (Gallier, 2015, [10]) A Riemannian manifold is a real smooth manifold $(M, g)$ equipped with an inner product $g_p$ at each point $p$ of its tangent space.

In a coordinate vector field $(\frac{\partial}{\partial x^i})$, $1 \leq i \leq n$ in the local neighborhood around point $p$. Let $u, v \in T_pM$ with

$$u = \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i}|_p \quad \text{and} \quad v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_p$$

Then $g_p(u, v) = \sum_{i,j} g_{ij}(p) u^i v^j$ where

$$g_{ij}(p) = g \left( \frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p \right)$$

A Riemannian metric can be written as $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$ or shortly $g = \sum_{i,j} g_{ij} dx^i dx^j$. Equipped with such metric, the differentiable manifold $(M, g)$ is a Riemannian manifold.

A geodesic is locally the shortest path in a Riemannian manifold.
Definition 5. (Gallot et al. 1990 [11]) let \((M,g)\) be a Riemannian manifold, and \(\frac{D}{dt}\) be the covariant derivative associated to \(g\) along \(\gamma\) also denoted by \(\nabla_{\gamma'}\). A smooth parametrized curve \(\gamma\) in \(M\) is a geodesic iff \(\frac{D}{dt}\gamma' = 0\).

Below we state the local existence and uniqueness of a geodesic.

Definition 6. (Petersen, 2006 [25]) Let \(m_0 \in M\). There exists an open set \(m_0 \in U\), and \(\epsilon > 0\) such that, for \(m \in U\) and \(q \in T_m M\) with \(|q| < \epsilon\), there is a unique geodesic \(c_q : ]-1 : 1[ \to M\) with \(c_q(0) = m\) and \(c_q'(0) = q\).

Definition 6 indicates that locally in a Riemannian manifold \(M\) around a point \(m\), given the tangent vector in \(m\), there is a unique geodesic.

Definition 7. (Gallier, 2015, [10]) Let \(m_0 \in M\). There exists a neighborhood \(U\) of \(m_0\) and \(\epsilon > 0\) such that, for any \(m,p \in U\), there is a unique geodesic \(c\) of length less than \(\epsilon\) from \(m\) to \(p\).

Definition 7 implies that there exists a unique geodesic that connects any two points in a local neighborhood of any Riemannian manifold.

Definition 8. (Gallier, 2012 [9]) Let \((M,g)\) be a Riemannian manifold. For every point, \(p \in M\), there is an open subset, \(W \subset M\), with \(p \in W\) and a number \(\epsilon > 0\), so that any two points \(q_1, q_2\) of \(W\) are joined by a unique geodesic of length \(< \epsilon\). For any other piece-wise smooth path, \(\omega\), joining \(q_1\) and \(q_2\), we have

\[
\int_0^1 \|\gamma'(t)\| \, dt \leq \int_0^1 \|\omega'(t)\| \, dt
\]

where the equality can hold only if the images \(\omega([0,1])\) and \(\gamma([0,1])\) coincide. Thus, \(\gamma\) is the shortest path from \(q_1\) to \(q_2\).

Given Definition 8 we know that locally, the shortest path between any two points in a Riemannian manifold is a geodesic.
We present the definition of isometric embedding as later we will prove an isometric embedding from a graph to a Riemannian manifold is not always possible. If we view Riemannian manifolds as metric spaces, we have the following definition.

**Definition 9.** (Gallot et al. 1990 [11]) Let \((M, g)\) be a connected Riemannian manifold. We define, for \(x, y \in M\), \(d(x, y)\) as the infimum of the lengths of all piece-wise \(C^1\) curves from \(x\) to \(y\). Then \(d\) is a distance in \(M\) which gives back to the topology of \(M\).

With any connected graph \(G = (V, E)\) one can associate a metric \(d_G : V \times V \rightarrow N\) by defining \(d_G(v, w)\), for \(v, w \in V\) to be the shortest distance between \(v\) and \(w\) in the graph. We say \(G'\) is an isometric (distance preserving) subgraph of \(G\) if, for all vertices \(v\) and \(w\), \(d_{G'}(v, w) = d_G(v, w)\). In our case, we embed a graph \(G\) to a Riemannian manifold \(M\). \(d_G\) is the metric space of graph \(G\), \(d_M\) is the metric space of \(M\) and \(d_M(v, w)\) is the geodesic distance between \(v\) and \(w\). If our mapping is isometric, we have \(d_G = d_M\) (distance preserving).

**Definition 10.** (Lee, 2005 [16]) A Riemannian manifold \((M, g)\) is said to have constant curvature if it is a simple-connected, complete Riemannian manifold with constant sectional curvature.

Gauss’s Theorema Egregium is a fundamental theorem in differential geometry. Informally, the theorem states that the curvature of a surface can be determined entirely by measuring distances along paths in the surface. That is, the curvature depends on the shortest path on the surface. On the other hand, the curvature reflects how shortest path is like in the surface. In the multiple-unicast problem, we assume each link has unit capacity. The cost depends on the shortest path between two nodes. Xiahou et al. [32] in their study prove that in Euclidean space, the multiple-unicast conjecture is true. Euclidean space is a constant 0 curvature 2-D Riemannian manifold. It is interesting to study the multiple-unicast using other 2-D Riemannian manifolds, including the ones with a constant curvature.
**Definition 11.** (Killing Hopf theorem [14], [15]) The universal cover of a manifold of constant Gaussian curvature is one of the model spaces: sphere (Gaussian curvature positive), plane (Gaussian curvature zero), and hyperbolic manifold (Gaussian curvature negative).

Some properties of 2-D constant curvature Riemannian manifolds will be used in the proofs in the later chapters.
Chapter 3

Related Work

Ahlswede et al. [3] introduced the fundamental idea of network coding, which enables nodes sending out packages as combination of local packets and packets received from upstream neighbors in the network. Potential benefits of network coding are found in network capacity, transmission efficiency, algorithm complexity, network security, and beyond.

Li et al. [19] focus on the direction of qualifying the potential of network coding in improving throughput of data transmission. A natural question here is how large the improvement can be due to the introduction of network coding. Their work proves that in an undirected network, the ratio of achievable throughput with and without network coding is upper-bounded by a constant 2. It proves that this result is valid in not only fraction routing but also half-integer routing. Research along this direction has been extended Internet-like bidirected networks, networks with integral link capacities and flow rates, and hypergraph network models. Li et al. also provide evidences to show that the achievable throughput and the coding advantage are irrelevant from of the selection of the information source within a unicast, broadcast, or multicast communication group. They also show that while network coding may not substantially improve the absolute maximum throughput, it does bring a significant benefit in practice, by changing the computational complexity of computing the maximum throughput from NP-hard to P.

Li and Li [18] study network coding for multiple-unicast sessions. They discover that for directed multiple-unicast networks, the advantage is unbounded when equipped with network coding. However, for the case of undirected networks, the coding advantage is not determined. Their work proposes the multiple-unicast network coding conjecture which states that for multiple unicast sessions in an undirected network, network coding can not outper-
form conventional routing. In the throughput domain, the conjecture is that a throughput vector is feasible for multiple-unicast sessions in an undirected network with network coding if and only if it is feasible with routing. It can be translated into an equivalent statement in the cost domain: in a cost-weighted undirected network \((G, \omega)\), let \(f\) be the flow vector of a network coding solution for \(k\) independent unicast sessions with throughput vector \(r\), then 
\[
\sum_{e}(f(e)\omega(e)) \geq \sum_{i}(d_{i}r_{i}).
\]
This open problem has attracted a series of studies, and has yet remained open in the past 12 years.

Harvey et al. \[13\] study the Okamura-Seymour example, which is a five-node graph that contains 4 pairs of terminals. Assuming each edge has unit capacity. They show that the maximum concurrent flow in the graph is \(3/4\). Using techniques from graph theory and information inequalities, they prove that the network coding rate in the network is also \(3/4\). Thus the coding advantage in this case of multiple-unicast network is 1. This is despite the fact that there is a gap between the multicommodity flow rate and the cut set bound in the Okamura-Seymour network.

Al-Bashabsheh and Yongacoglu \[4\] further explore another example of multiple-unicast network, Hu’s 3-commodity network \(N_{2}\). They show that the network coding rate is \(8/7\) by applying input and output inequality. They further show that the network can achieve throughput \(8/7\) with a fractional routing scheme. Thus network coding does not have any advantage over routing, verifying the multiple-unicast conjecture in Hu’s 3-commodity network.

Xiahou et al. \[32\] study the multiple-unicast network coding conjecture from a geometry perspective. Their framework consists of four steps. First it shows one can translate the conjecture from throughput domain to cost domain by LP duality. In the second step, by embedding graphs into high dimensional Euclidean space, the conjecture is then translated from graph domain into space domain. In step 3, the problem is reduced from a high dimension space into 1-D space. Lastly, a direct proof of the conjecture in 1-D scenario
concludes the framework. Since the upper bound for the distortion for embedding general
graphs into Euclidean space is $O(\log n)$, the coding advantage for general graph is thus upper
bounded by $O(\log k)$. A potential approach to reduce the distortion suggests embedding
dographs into Riemannian manifolds instead of Euclidean space.

Yin et al. [33] propose a reduction method to study the multiple-unicast conjecture
by applying information inequalities and graph decomposition. They characterized a new
class of networks based on the relations between cut-sets and source-receiver paths. Their
techniques turn out to be sufficient to prove that the multiple-unicast conjecture is true for
all networks with up to 6 nodes. They also prove that network coding is equivalent to routing
in unit length undirected network for up to 7 nodes and 3 unicast pairs.

Braverman et al. [6] recently study network coding in undirected graphs, and prove that
network coding has either no coding advantage or near-maximum advantage over routing.
Given two graphs $G_1$ and $G_2$ with small gaps $1 + \epsilon_1$ and $1 + \epsilon_2$ between routing rate and
network coding rate, they create a graph $G$ with a gap of $(1 + \epsilon_1)(1 + \epsilon_2)$ with increased
size, using a form of graph product. A graph with an even larger gap can be constructed by
tensoring $G$ with itself and then repeatedly tensoring the previous iteration with itself. As
a result, an infinity family of graphs $G'$ that achieves a gap of $O(\log |G'|)^c$ for some constant
c $< 1$. By such approach, they show even a graph with a small coding advantage can be
amplified to a near-maximum possible advantage. The following statements are equivalent
to their findings. (i) If there exists a counter example to the multiple-unicast conjecture,
then there exists a counter example in which the coding advantage approaches infinity. (ii)
If there is a constant upper bound of the coding advantage for multiple-unicast network
coding in undirected networks, then the multiple-unicast conjecture is true. In other words,
the conjecture is equivalent to claiming that there is a constant upper bound of the coding
advantage.
Chapter 4

Almost Isometric Embedding of Graphs into Riemannian Manifolds

In this chapter, we aim to embed general undirected graphs into Riemannian manifolds. Xiahou et al. [32] showed the multiple-unicast conjecture is true in Euclidean space. Euclidean space is a subset of Riemannian manifolds, and has a distortion of embedding of $O(\log n)$. We intend to discover a method to reduce the distortion to a constant 1 (isometrical embedding), or close enough to 1, such that we can use the embedding to help study the multiple-unicast conjecture using a geometric framework similar to that proposed by Xiahou et al.. First we show that not all undirected graphs can be isometrically embedded into a Riemannian manifold. However, fortunately, we proceed to show that an embedding with a distortion arbitrarily close to 1 is possible, and consequently, if the multiple unicast conjecture is true in Riemannian manifolds then it is true in undirected networks as well.

4.1 Non-existence of an Isometric Embedding

**Theorem 1.** Not all undirected graphs can be isometrically embedded into a Riemannian manifold.

**Proof.** We prove by way of contradiction.

Consider the undirected graph in Fig. 4.1. Assume this undirected graph $G = (V, E)$ can be isometrically embedded into some Riemannian manifold $M$. In $M$, we have $OA$ to be the shortest path between $O$ and $A$, similarly for $AB$ and $AC$. Since the shortest path $d_{OB} = d_{OA} + d_{AB}$ in $G$, and the distances are preserved in $M$, we have a shortest path from $O$ to $B$ through $OA$ and $AB$. Furthermore, a shortest path from $O$ to $A$ passes through $OA$
Figure 4.1: An example of an undirected graph that can not be isometrically embedded into a Riemannian manifold.

and $AC$. The two shortest paths overlap partially in $OA$. Let $A'$ be the last point where $OB$ and $OC$ intersect. There exists a small neighborhood around $A'$ with a radius $\epsilon > 0$ arbitrarily close to 0. It intersects with $OA'$ at $O'$, $OB$ at $B'$ and $OC$ at $C'$. $B'$ and $C'$ are two distinct points as $A'$ is the last intersection point. The path $OA' - A'B$ is the shortest path between $O$ and $B$, and $O'A' - A'B'$ is the shortest path between $O'$ and $B'$, otherwise it has a contradiction that $OO' - O'B' - B'B$ would have a smaller distance.

From Definition 8, we learn that $O'A' - A'B'$ is a geodesic since it is locally the shortest path between $O'$ and $C'$. For the same reason, $O'A' - A'C'$ is a geodesic. $O'A' - A'B'$ being the shortest path implies that $O'A'$ is the local shortest path and a geodesic. According to Definition 7, there is a unique geodesic $\gamma_{O'A'}$ as the shortest path between $O'$ and $A'$.

Two geodesics overlap in $O'A'$. They share the same value of $\gamma_{O'A'}$ at the point $O'$. $O'A' - A'B'$ is denoted by $c_1(t)$, and $O'A' - A'C'$ is denoted by $c_2(t)$.

By Definition 6, we know that $c_1(t)$ and $c_2(t)$ satisfy

$$\frac{D}{dt}c'_1 = 0 \text{ and } \frac{D}{dt}c'_2 = 0$$

$\frac{D}{dt}\gamma' = 0$ is a second order ODE. By Cauchy–Lipschitz Theorem, given $\gamma(0) = O'$ and
\[ \gamma'(0) = \gamma'_{O'A}(O'), \] there is a unique solution for \( \gamma \). This implies that two geodesics which pass the same point with the same tangent value must be the same geodesic. Geodesics \( c_1 \) and \( c_2 \) overlap in \( O'A' \), and share the same tangent value at \( O' \). There is a unique solution for \( \frac{D}{dt} \gamma' = 0 \). Therefore, \( c_1 = c_2 \), namely, \( O'A' - A'B' \) and \( O'A' - A'C' \) are the same path, and \( B' \) and \( C' \) are the same point. This contradicts our assumption that \( A' \) is the last intersection point. Thus the undirected graph we consider can not be isometrically embedded into any Riemannian manifold. In conclusion, not all undirected graphs can be isometrically embedded into a Riemannian manifold.

4.2 Smooth Connection to Spheres

An embedding of an undirected graph \( G = (V, E) \) into a Riemannian manifold \( M \) involves assigning a point in \( M \) to each node \( u \in V \). We model each node as an arbitrary point in its own corresponding 2-D sphere for the reason that the connections from other portions to the spheres can come from any possible direction. A key step of our embedding is to find a way to smoothly connect to those spheres as a necessary condition for the entire construction resulting in a Riemannian manifold. Thus we begin to construct a piecewise smooth curve \( s(x) \). Then we rotate \( s(x) \) into a manifold and prove this manifold is indeed a Riemannian manifold. We give the definitions of smooth function and smooth connection.

**Definition 12.** (Weisstein, 2016) A smooth function is a function that has continuous derivatives of all orders everywhere in its domain.

**Definition 13.** Two functions are smoothly connected if and only if they have the same derivatives of all orders at their connecting point.

In order to find such a smooth connection, we start from the basics and seek a smooth curve connecting to a circle as shown in Fig. 5.1. A tangent line \( L(x) \) is not smoothly connected to a circle \( R(x) \) since the second derivative of \( L(x) \) is 0, while for circle \( R(x) \) at
the tangent point is not. A smooth connection between two curves requires equal derivatives for $C^\infty$.

We consider a smooth function

$$\varphi(x) = \begin{cases} 
  e^{-\frac{1}{x}} & x > 0 \\
  0 & x \leq 0 
\end{cases} \quad (4.1)$$

where $\varphi(x)$ is a smooth piecewise curve, and the $n^{th}$ derivatives are always 0 at $x = 0$, according to Mei [21]. We can see that $\varphi(x)$ smoothly connects to a line. With the help of $\varphi(x)$, we construct a curve

$$f(x) = (1 - \alpha(x))\varphi(x) + \alpha(x)R(x) \quad (4.2)$$

We aim to utilize $f(x)$ as an intermediate curve that smoothly connects to both $L(x)$ and $R(x)$. It requires $\alpha(x)$ to approach 0 near $x = 0$ and 1 near $\xi \in (0, r]$. Besides, for any positive integer $n$, we want the $n^{th}$ derivative of $\alpha(x)$ to be always 0 at $x = 0$ such that $f(x)$ smoothly connects to $L(x)$. The $n^{th}$ derivative of $\alpha(x)$ is also required to be constant 0 at $x = \xi$ such that $f(x)$ also smoothly connects to $R(x)$.

**Theorem 2.** Let $L(x) = 0, x \leq 0$ and $R(x)$ be a portion of a circle with $x \geq \xi$, where $\xi$ is a positive number. $f(x)$ as defined in Eq. 4.2 is a smooth function that connects to both $L(x)$ and $R(x)$. If $\alpha(x)$ satisfies
(i) \( \alpha(0) = 0 \) such that \( f(0) = \varphi(0) = 0 \).

(ii) \( \alpha(\xi) = 1 \) such that \( f(\xi) = R(\xi) \).

(iii) The \( n \)th derivative of \( \alpha(x) \) is 0 at \( x = 0 \).

(iv) The \( n \)th derivative of \( \alpha(x) \) is 0 at \( x = \xi \).

then \( f(x) \) smoothly connects to both \( L(x) \) and \( R(x) \).

\textbf{Proof.} In order to show \( f(x) \) smoothly connects both \( L(x) \) and \( R(x) \), we first calculate the \( n \)th derivative of \( f(x) \).

\[
\frac{\partial^n f(x)}{\partial x^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k (1 - \alpha(x))}{\partial x^k} \frac{\partial^{n-k} \varphi(x)}{\partial x^{n-k}} + \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k \alpha(x)}{\partial x^k} \frac{\partial^{n-k} R(x)}{\partial x^{n-k}}
\]

Assume \( \alpha(x) \) has the following properties:

\begin{align*}
\alpha(0) &= 0 \quad (4.3) \\
\alpha(\xi) &= 1 \quad (4.4) \\
\frac{\partial^k \alpha(0)}{\partial x^k} &= 0 \text{ for } k = \{1, 2, \ldots\} \quad (4.5) \\
\frac{\partial^k \alpha(\xi)}{\partial x^k} &= 0 \text{ for } k = \{1, 2, \ldots\} \quad (4.6)
\end{align*}

Also for \( \varphi(x) \),

\[
\frac{\partial^k \varphi(0)}{\partial x^k} = 0 \text{ for } k = \{0, 1, 2, \ldots\} \quad (4.7)
\]

We apply Eq. 4.3, Eq. 4.7 and then for \( n \in \{0, 1, 2, \ldots\} \)

\[
\frac{\partial^n f(0)}{\partial x^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k (1 - \alpha(0))}{\partial x^k} \frac{\partial^{n-k} \varphi(0)}{\partial x^{n-k}} + \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k \alpha(0)}{\partial x^k} \frac{\partial^{n-k} R(0)}{\partial x^{n-k}} = 0
\]
and
\[
\frac{\partial^n f(\xi)}{\partial x^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k (1 - \alpha(\xi))}{\partial x^k} \frac{\partial^{n-k} \varphi(\xi)}{\partial x^{n-k}}
\]
\[
+ \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k \alpha(\xi)}{\partial x^k} \frac{\partial^{n-k} R(\xi)}{\partial x^{n-k}} = \frac{\partial^n R(\xi)}{\partial x^n}
\]

For both \(L(x)\) and \(R(x)\), \(f(x)\) has continuous derivatives up to infinity order at their connecting point. Therefore, if \(\alpha(x)\) satisfies Eq. 4.3- Eq. 4.6, \(f(x)\) smoothly connects to both \(L(x)\) and \(R(x)\). \(\square\)

We can find

\[
\beta(x) = \frac{\varphi(x)}{\varphi(x) + \varphi(\xi - x)}
\]

such that

\[
\alpha(x) = \begin{cases} 
0 & x \leq 0 \\
\beta(x) & 0 < x \leq \xi \\
1 & \xi \leq x 
\end{cases}
\]
is a smooth function in \(\mathbb{R}\), and satisfies the requirements Eq. 4.3- Eq. 4.6

Therefore,

\[
\begin{aligned}
s(x) &= \begin{cases} 
0 & x \leq 0 \\
(1 - \alpha(x))\varphi(x) + \alpha(x)R(x) & 0 < x \leq \xi \\
R(x) & \xi \leq x 
\end{cases}
\end{aligned}
\]
is a smooth function in \(\mathbb{R}\) as illustrated in Fig. 4.3. In fact, \(R(x)\) could be any smooth function in \(\mathbb{R}\).

Further we can extend our finding to a smooth intermediate curve that smoothly connects two arbitrary smooth curves.

**Theorem 3.** For two arbitrary smooth functions \(v(x), x \leq 0\) and \(R(x), x \geq \xi\) in \(\mathbb{R}\),

\[
\begin{aligned}
s(x) &= \begin{cases} 
v(x) & x \leq 0 \\
(1 - \alpha(x))\varphi(x) + \alpha(x)R(x) + v(x) & 0 < x \leq \xi \\
R(x) + v(x) & \xi \leq x 
\end{cases}
\end{aligned}
\]
Figure 4.3: A sketch of $s(x)$ which is an intermediate curve smoothly connecting a line and a circle.

is also a smooth function in $\mathbb{R}$, where $\xi$ is a positive number.

**Proof.**

$$
\lim_{x \to 0^-} s(x) = v(0)
$$

$$
\lim_{x \to 0^+} s(x) = (1 - \alpha(0))\varphi(0) + \alpha(0)R(0) + v(0) = v(0)
$$

Thus $s(x)$ is continuous at $x = 0$.

For $k = \{1, 2, \ldots\}$,

$$
\lim_{x \to 0^-} \frac{\partial^k s(x)}{\partial x^k} = \frac{\partial^k (v(0))}{\partial x^k}
$$

$$
\lim_{x \to 0^+} \frac{\partial^k s(x)}{\partial x^k} = \frac{\partial^k ((1 - \alpha(0))\varphi(0) + \alpha(0)R(0) + v(0))}{\partial x^k}
$$

$$
= \frac{\partial^k ((1 - \alpha(0))\varphi(0) + \alpha(x)R(0))}{\partial x^k} + \frac{\partial^k (v(0))}{\partial x^k} = \frac{\partial^k (v(0))}{\partial x^k}
$$

Thus $s(x)$ is smooth at $x = 0$.

$$
\lim_{x \to \xi^-} s(x) = (1 - \alpha(\xi))\varphi(\xi) + \alpha(\xi)R(\xi) + v(\xi) = R(\xi) + v(\xi)
$$

$$
\lim_{x \to \xi^+} s(x) = R(\xi) + v(\xi)
$$
Thus \( s(x) \) is continuous at \( x = \xi \).

For \( k = \{1, 2, \ldots\} \),
\[
\lim_{x \to \xi^-} \frac{\partial^k(s(x))}{\partial x^k} = \frac{\partial^k((1 - \alpha(\xi))\varphi(\xi) + \alpha(\xi)R(\xi) + v(\xi))}{\partial x^k}
\]
\[
= \frac{\partial^k((1 - \alpha(\xi))\varphi(\xi) + \alpha(\xi)R(\xi))}{\partial x^k} + \frac{\partial^k(v(\xi))}{\partial x^k}
\]
\[
= \frac{\partial^k(R(\xi) + v(\xi))}{\partial x^k}
\]
\[
\lim_{x \to \xi^+} \frac{\partial^k(s(x))}{\partial x^k} = \frac{\partial^k(R(\xi) + v(\xi))}{\partial x^k}
\]

It shows that \( s(x) \) is smooth at \( x = \xi \). Therefore, \( s(x) \) is a smooth function in \( \mathbb{R} \).

4.3 Almost Isometric Embedding of Undirected Graphs into Riemannian Manifolds

In this section, we present our approach to embed an undirected graph \( G = (V, E) \) into a Riemannian manifold \( M \). The details are described as follows.

We start with a basic scenario. Consider mapping a basic undirected graph \( G_c = (V, E) \), two vertices \( u \in V \) and \( v \in V \) connected by edge \( e \in E \), into a Riemannian manifold. Two vertices \( u \) and \( v \) are modeled as arbitrary points \( u' \) and \( v' \) in their own corresponding 2-D sphere. The edge \( e \) is modeled as a shortest path between the two points. The general idea is to construct a smooth curve which will be rotated into a manifold.

As illustrated in Fig. 4.4, we begin to construct a smooth curve containing two partial circles with radius \( r \) and a Cosine curve. We choose circles since they can be rotated into 2-D spheres. Also, the Cosine curve is chosen because the length of the curve can be controlled by changing its frequency. We utilize Theorem 3 to find two intermediate smooth curves to smoothly connect two partial circles with the Cosine curve. There are two constraints in our model. (i) The amplitude of the Cosine function \( m \) should be much smaller than \( r \) to ensure no Cosine portions overlap when later combined into a more sophisticated manifold. (ii) The
radius $r$ should be much smaller than the length of the edge, that is, $m << r << |e|$. To ensure both constraints, we set the radius $r$ to be a constant arbitrarily close to 0 and $m$ to be $r^2$.

Next, we rotate the smooth curve along the line connecting the two centers of the partial circles into a basic manifold which contains two partial spheres, two manifolds from rotating the intermediate curves and one manifold from rotating the Cosine curve. We use $M_c$ to denote the basic manifold in our construction. To embed more sophisticated graphs, we can first embed each edge with its two vertices into a basic manifold. Recall that each sphere portion has a constant $r$ radius which is arbitrarily close to 0. Thus we can combine the basic manifolds into a more complicated manifold by merging the sphere portions which correspond to the same node.

The rest of this section is organized as follows. For the purpose of our study, we first need to prove our construction above indeed results in a Riemannian manifold. Secondly, we will explore and find the shortest path between the two points $u'$ and $v'$ for the basic case $M_c$. At the end, the distortion of this graph embedding approach will be analyzed for the basic topology as well as for the general cases.

4.3.1 $M_c$ is a Riemannian manifold

The fundamental step is to show the basic manifold $M_c$ we constructed above is a Riemannian manifold. Following the definition of Riemannian manifolds in Definition 4, we begin by showing that (i) $M_c$ is a smooth manifold, and secondly prove that (ii) $M_c$ is equipped with
a metric tensor that varies smoothly in $T_xM_c$.

**Theorem 4.** (Gallier, 2012 [9]) Let $U \subseteq \mathbb{R}^{(n-k)+k}$ be an open set, and let $F: U \to \mathbb{R}^{n-k}$ be a $C^\infty$ differentiable mapping. Let $M \subseteq \mathbb{R}^{(n-k)+k}$ such that $M \cap U$ is the zero-locus of $F$ in $U$; that is
\[ M \cap U = \{ z \in U \mid F(z) = 0 \} \]

$M \cap U$ is the graph of a smooth function $f$, that is, a collection of all ordered pairs $(x, f(x))$. If $DF|_z$ is onto for every $z \in M \cap U$, then $M \cap U$ is a smooth manifold of dimension $k$ embedded in $\mathbb{R}^n$. If every $z \in M$ is in such a $U$ (i.e., it comes with a function $F: U \to \mathbb{R}^{n-k}$ so that $U \cap M$ is the zero-locus of the function $F$), then $M$ is a k-D manifold.

**Theorem 5.** $M_c$ is a smooth manifold.

**Proof.** In our construction, we have a piecewise smooth function $s(x) > 0$ as shown in Fig. 4.4. We construct $M_c$ by rotating the curve along with the $x$ axis. $M_c$ resides in $\mathbb{R}^3$. We have
\[ M_c := \{ (x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 - (s(x))^2 = 0 \} \]

$M_c$ is a non-empty set. We define a continuously differentiable function $F$ for which $M$ is the zero-locus.
\[
F : \mathbb{R}^3 \to \mathbb{R}^1 \text{ given by } F: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (y^2 + z^2 - (s(x))^2)
\]

The zero-locus $F^{-1}(0)$ represents $M_c$. We have
\[
DF = [2s'(x)s(x), 2y, 2z]
\]

Given Theorem 4, we need to assure $DF$ has rank 1 for any point in $M_c$. Since $y^2 + z^2 - (s(x))^2 = 0$, and $s(x)$ has positive values in the domain except for the two end points, $y$ and $z$ can never be 0 at the same time except for the two end points. As to the two end points,
without losing generality, we consider the end point with \( x > 0 \). The end point \( x_e \) locates in a portion of circle. The corresponding smooth curve is

\[
s(x) = \sqrt{r^2 - (x - k)^2}
\]

where \( k > r \) is some positive constant. Thus

\[
s'(x) = \frac{k - x}{\sqrt{r^2 - (x - k)^2}}
\]

\[
2s'(x)s(x) = 2(k - x)
\]

\[
2s'(x_e)s(x_e) = 2(k - x_e) = 2(k - r - k) = -2r
\]

It can be concluded that at the end points with \( x > 0 \), \( DF = [-2r, 0, 0] \) is not \( \vec{0} \), and has rank 1. Similarly, we can draw the same conclusion for the other end point.

Thus \( DF \) is not \( \vec{0} \) for any point in \( M_c \). Besides, \( f = y^2 + z^2 - (s(x))^2 \) is a smooth function. Therefore, \( M_c \) is a smooth manifold.

\[\square\]

**Theorem 6.** \( M_c \) is a Riemannian Manifold.

**Proof.** By Definition 1, we know that a Riemannian manifold is a real smooth manifold equipped with a metric tensor that varies smoothly in the tangent space. In Theorem 5, we prove that \( M_c \) is a smooth manifold. Below we construct the metric tensor of \( M_c \) in order to show \( M_c \) is a Riemannian manifold.

\[
\gamma = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s(t)\cos(\theta) \\ s(t)\sin(\theta) \\ t \end{pmatrix}
\]

with

\[
\gamma'_{\theta} = \begin{pmatrix} -s(t)\sin(\theta) \\ s(t)\cos(\theta) \\ t \end{pmatrix}
\]
\[ \vec{\gamma}'_t = \begin{pmatrix} s'(t)\cos(\theta) \\ s'(t)\sin(\theta) \\ 1 \end{pmatrix} \]

Thus
\[
\vec{\gamma}_\theta \cdot \vec{\gamma}_\theta = (-s(t)\sin(\theta))^2 + (s(t)\cos(\theta))^2 + t^2
= s^2(t) + t^2
\]
\[
\vec{\gamma}_t \cdot \vec{\gamma}_t = (s'(t)\cos(\theta))^2 + (s'(t)\sin(\theta))^2 + 12
= s'^2(t) + 1
\]
\[
\vec{\gamma}_\theta \cdot \vec{\gamma}_t = (-s(t)\sin(\theta)) \times (s'(t)\cos(\theta)) + (s(t)\cos(\theta)) \times (s'(t)\sin(\theta)) + t = t
\]

We have the metric tensor
\[
g = \begin{pmatrix} \vec{\gamma}_\theta \cdot \vec{\gamma}_\theta & \vec{\gamma}_\theta \cdot \vec{\gamma}_t \\ \vec{\gamma}_\theta \cdot \vec{\gamma}_t & \vec{\gamma}_t \cdot \vec{\gamma}_t \end{pmatrix} = \begin{pmatrix} s^2(t) + t^2 & t \\ t & s'^2(t) + 1 \end{pmatrix}
\]
where \(s(x)\) is a \(c^\infty\) smooth curve in the domain. \(g\) is positive definite and varies smoothly.

\(M_c\) is a smooth manifold, and is equipped with a metric tensor \(g\) in the tangent space that varies smoothly from point to point in \(M_c\). Therefore, \(M_c\) is a Riemannian manifold.

### 4.3.2 Broken Geodesic

Definition 7 asserts that any sufficiently small segment of a geodesic is minimal, which implies that a geodesic is locally the shortest path. However, the global shortest path is the infinitum of all possible curves connecting two points but not necessarily a geodesic. We characterize the global shortest path in \(M_c\): a global shortest path in a connected manifold is a broken geodesic.
**Definition 14.** (Gallier, 2015 [10]) A broken geodesic is a piecewise smooth curve where each curve segment is a geodesic.

**Theorem 7.** (Gallier, 2015 [10]) A Riemannian manifold, \((M, g)\), is connected if and only if any two points of \(M\) can be joined by a broken geodesic.

Intuitively, \(M_c\) is a connected manifold, thus any two points of \(M_c\) can be joined by a broken geodesic. Further, we prove the following theorem.

**Theorem 8.** For any two points in \(M_c\), the shortest path is a broken geodesic.

*Proof.* We prove by way of contradiction.

Consider two points \(u'\) and \(v'\) in a Riemannian manifold. Assume the shortest path between \(u'\) and \(v'\) is not a broken geodesic. Let \(c\) denote the shortest path from \(u'\) to \(v'\) with some local segment that is not a geodesic. In such a local segment \(\bar{c}\), we can find a geodesic as the unique shortest path for two arbitrary points according to Definition 8. The local segment \(\bar{c}\) can be replaced with a geodesic to reduce the length of \(\bar{c}\). By such approach, we replace all the non-geodesic segments in \(c\) with geodesics so the path becomes a broken geodesic, and the length of \(c\) decreases. In conclusion, that \(c\) is not the shortest path gives a contradiction. Therefore, for any two points in \(M_c\), the shortest path is a broken geodesic.

### 4.3.3 The Shortest Path

We next examine the shortest paths in the embedded Riemannian manifold \(M_c\). We start from the basic case. Recall that the basic case \(M_c\) is constructed by rotating a piecewise smooth curve \(s(x)\) which consists of several parts: two partial circles, a Cosine curve and two smooth intermediate curve. For convenience, two manifolds from rotating partial circles are denoted by \(M_{r_1}\) and \(M_{r_2}\), the manifold from rotating Cosine curve as \(M_{\cos}\) and the two manifolds from rotating intermediate curves as \(M_{i_1}\) and \(M_{i_2}\). Recall that we set the radius

\[33\]
of partial circles, \( r \), to be a constant arbitrarily close to 0, and the amplitude of the Cosine curve, \( m \), to be \( r^2 \). Thus \( m \ll r \ll |e| \) which ensures manifolds would not overlap in constructions for general undirected graphs. The upper bound for the length of the shortest path between \( u' \) and \( v' \) in \( M_c \) is denoted by \( U_{u'v'} \). Similarly, We use \( L_{u'v'} \) to represent the lower bound for the length of the shortest path between \( u' \) and \( v' \) in \( M_c \). We aim to find a precise enough range for the length of the shortest path. The idea is that by controlling the frequency of the Cosine curve, we can adjust the length of the Cosine curve, thus changing the value of the upper bound \( U_{u'v'} \) and lower bound \( L_{u'v'} \) for the following inequality to hold.

\[
L_{u'v'} \leq l_{u'v'} \leq U_{u'v'} \leq d_{uv} \leq (1 + \epsilon)L_{u'v'} \leq (1 + \epsilon)l_{\gamma_{u'v'}}
\]

where \( \gamma_{u'v'} \) is the shortest path between points \( u' \) and \( v' \) in \( M_c \), \( l_{\gamma_{u'v'}} \) is the length of \( \gamma_{u'v'} \) and \( d_{uv} \) is the shortest path between the corresponding nodes \( u \) and \( v \) in the graph \( G_c \).

The distortion of the embedding is

\[
\frac{d_{uv}}{l_{\gamma_{u'v'}}} \leq \frac{(1 + \epsilon)L_{u'v'}}{L_{u'v'}} = 1 + \epsilon
\]

If we can find an \( \epsilon \) which is positive and arbitrarily close to 0, we can reduce the distortion of the embedding to arbitrarily close to 1.

Below we study the lower bound \( L_{u'v'} \) for the length of the shortest path between \( u' \) and \( v' \) in \( M_c \). From Theorem 8, we know that the global shortest path is a broken geodesic. For two points in \( M_{r_1} \) and \( M_{r_2} \), the shortest path must go through the Cosine portions \( M_{cos} \) and the intermediate portions \( M_{i_1} \) and \( M_{i_2} \). We can divide the global shortest path into 5 parts in corresponding 5 portions, and each piece should be a broken geodesic as well. To seek the lower bound \( L_{u'v'} \), we can simply find the lengths of the shortest paths in each portion, and sum them up. The length of the shortest path between two end points in \( M_{cos} \) is equal to the length of the Cosine curve. The Cosine curve is denoted by \( c_{cos} \) and its length by \( l_{cos} \). We find a lower bound for the length of the shortest path to be \( l_{cos} \), by setting the lengths of the portions of shortest paths in \( M_{r_1} \), \( M_{r_2} \), \( M_{i_1} \) and \( M_{i_2} \) to be 0. As later we prove the
length of the portions of shortest path in \( M_{r_1}, M_{r_2}, M_{i_1} \) and \( M_{i_2} \) are much smaller than \( l_{\cos} \), the lower bound is tight enough for our purpose.

It remains to find the upper bound \( U_{u'v'} \). The idea is similar to the calculation of the lower bound \( L_{u'v'} \). By adding up the upper bounds of lengths of broken geodesics in \( M_{\cos}, M_{r_1}, M_{r_2}, M_{i_1} \) and \( M_{i_2} \), we are able to obtain an upper bound of the length of the shortest path.

First we show the lengths of the broken geodesics in \( M_{r_1}, M_{r_2}, M_{i_1} \) and \( M_{i_2} \) are arbitrarily close to 0. Since \( M_{r_1} \) is a partial sphere, the length of the shortest path between any two points in \( M_{r_1} \) with radius \( r \) is at most the length of the shortest path between two opposite poles. Thus the length of shortest paths in \( M_{r_1} \) are at most \( \pi r \) which is arbitrarily close to 0, similarly for \( M_{r_2} \). Let \( c_{i_1} \) and \( c_{i_2} \) denote the intermediate curves. \( l_{i_1} \) and \( l_{i_2} \) denote their lengths. Below we show the length of shortest path in \( M_{i_2} \) (rotated from the intermediate curve on the right hand as shown in Fig. 4.4) is at most the length of smooth intermediate curve \( c_{i_2} \) is smaller than \( r \), thus arbitrarily close to 0.

We first study the right portion \( s(x) \), denoted by \( \bar{s} \). We have

\[
\bar{s}(x) = \begin{cases} 
2r^2 + r^2 \cos(\beta x) & x \leq 0 \\
(1 - \alpha(x))\varphi(x) + \\
\alpha(x)(\sqrt{r^2 - (x - r)^2} - 2r^2 + r^2 \cos(\beta x)) + \\
2r^2 + r^2 \cos(\beta x) & 0 < x \leq r^2 \\
\sqrt{r^2 - (x - r)^2} & r^2 \leq x \leq 2r
\end{cases}
\]

where \( r \) is arbitrarily close to 0. Also,

\[
\varphi(x) = \begin{cases} 
e^{-\frac{1}{x}} & x > 0 \\
0 & x \leq 0
\end{cases}
\]
Figure 4.5: A sketch of $\bar{s}(x)$ which is a portion of the curve in Fig. 4.4.

$$\alpha(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{\varphi(x)}{\varphi(x) + \varphi(r^2 - x)} & 0 < x \leq r^2 \\
1 & r^2 \leq x 
\end{cases}$$

Fig. 4.5 shows an example of $\bar{s}$. From Theorem 3, we know that $\bar{s}$ is a smooth function in its domain $x \leq 2r$.

On $0 < x \leq r^2$,

$$\bar{s}(x) = (1 - \alpha(x))\varphi(x) + \alpha(x)\left(\sqrt{r^2 - (x - r)^2} - 2r^2 - r^2\cos(\beta x)\right) + 2r^2 + r^2\cos(\beta x)$$

We want to show that the length of smooth intermediate curve, $l_{i_2}$, is smaller than $r$, where

$$l_{i_2} = \int_0^{r^2} \sqrt{1 + s'(x)^2} \, dx$$

**Lemma 1.** $\bar{s}'(x)$ achieves its maximum at $x = r^2$ for $0 \leq x \leq r^2$.

**Proof.**

$$\bar{s}'(x) = (1 - \alpha(x))\varphi'(x) - \alpha'(x)\varphi(x) + \alpha'(x)(\sqrt{r^2 - (x - r)^2} - 2r^2 - r^2\cos(\beta x)) + \alpha(x)\left(\frac{r - x}{\sqrt{x(2r - x)}} + \beta r^2\sin(\beta x)\right) - \beta r^2\sin(\beta x)$$
Since $\alpha(x) \in [0, 1]$ and $\lim_{x \to 0} \alpha'(x) \to 0$, all items have value less or equal to 1 for $0 < x \leq r^2$ except for $\lim_{x \to 0} \frac{r-x}{\sqrt{x(2r-x)}} \to 0$. Therefore, $\alpha(x)\frac{r-x}{\sqrt{x(2r-x)}}$ dominates the value of $s'(x)$. While when $x$ is close to 0, $\alpha(x)$ is exponentially close to 0 which dominates $\frac{r-x}{\sqrt{x(2r-x)}}$. Thus $\alpha(x)\frac{r-x}{\sqrt{x(2r-x)}}$ achieves its maximum at $x \to r^2$, where $0 \leq x \leq r^2$, and $r$ is arbitrarily close to 0. Therefore, $s'(x)$ achieves its maximum at $r^2$ for $0 \leq x \leq r^2$.

\[ \square \]

**Lemma 2.** The length of $c_{i_2}$ is smaller than $r$.

**Proof.** Given Lemma 1, we have

\[ l_{i_2} = \int_0^{r^2} \sqrt{1 + s'(x)^2} dx \leq r^2 \times \sqrt{1 + s'(r^2)^2} \]

Since $s(x)$ is a smooth curve, $s'(r^2)$ has the same value at $x = r_-^2$ and $x = r_+^2$. Thus

\[ s'(r^2) = \frac{r - r^2}{\sqrt{r^2(2r - r^2)}} \]

\[ l_{i_2} \leq r^2 \times \sqrt{1 + s'(r^2)^2} = r^2 \times \sqrt{1 + \left(\frac{r - r^2}{\sqrt{r^2(2r - r^2)}}\right)^2} \]

\[ = \sqrt{\frac{r^4}{2r - r^2}} \]

Additionally, for $r$ is arbitrarily close to 0,

\[ l_{i_2} \leq \sqrt{\frac{r^4}{2r - r^2}} < r \]

Therefore, the length of the smooth intermediate curve, $l_{i_2}$, is smaller than $r$, thus arbitrarily close to 0.

The piecewise smooth curve $s(x)$ is symmetric, thus $l_{i_1} = l_{i_2}$. We can conclude that the lengths of the piecewise geodesics in $M_{r_1}$, $M_{r_2}$, $M_{i_1}$ and $M_{i_2}$ are arbitrarily close to 0. We can find a tight enough lower bound of the shortest path $L_{uv'}$ which is the length of the Cosine curve, $l_{cos}$. 

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Lemma 3. An upper bound for the length of the shortest path between $u'$ and $v'$ in $M_c$ is $2\pi r + 2r + l_{\cos}$.

Proof. Consider a path $c_U$ of length $l_U$ passing $c_{\cos}, c_{i_1}, c_{i_2}$ and then connecting $u'$ and $v'$ in two ends. $c_U$ intersects the boundary between $M_{r1}$ and $M_{i1}$ at $\tilde{u}'$ and the boundary between $M_{r2}$ and $M_{i2}$ at $\tilde{v}'$. $u'$ and $\tilde{u}'$ are both in $M_{r1}$, meanwhile $v'$ and $\tilde{v}'$ are both in $M_{r2}$. The length of any curve connecting the two points $u'$ and $v'$ is larger than or equal to the length of the shortest path between $u'$ and $v'$. Thus we set $l_U$ as the upper bound $U_{u'v'}$. $l_U$ have several parts. The upper bounds of the partial lengths of $c_U$ in $M_{r1}$ and $M_{r2}$ are both $\pi r$. $c_U$ also contains $c_{i_1}, c_{i_2}$ and $c_{\cos}$. Recall that $l_{i_1} = l_{i_2}$. We have

$$U_{u'v'} = 2\pi r + 2l_{i_2} + l_{\cos}$$

$$< 2\pi r + 2r + l_{\cos}$$

where

$$l_{i_2} \leq \sqrt{\frac{r^4}{2r - r^2}} < r$$

as $r$ is arbitrarily close to 0. Thus we can find an upper bound for the length of the shortest path in $M_c$, which is

$$U_{u'v'} = 2\pi r + 2r + l_{\cos}$$

Theorem 9. Let $r$ be a constant arbitrarily close to 0 and $G_c = (V, E)$ be a basic undirected graph containing two vertices $u \in V, v \in V$ connected by an edge $e \in E$. The distortion of the graph $G_c$ embedding into the host manifold $M_c$ is arbitrarily close to 1.

Proof. Two vertices $u$ and $v$ in $G_c$ are modeled as arbitrary points $u'$ and $v'$ in $M_c$. Consider the length of the shortest path between $u'$ and $v'$. Recall that, the lower bound $L_{u'v'} = l_{\cos}$. The length of the Cosine curve $l_{\cos}$ can be controlled by its frequency so that $l_{\cos} = d_{uv}/(1+\epsilon)$,
where $\epsilon$ is a positive constant, and $d_{uv}$ is the length of the edge $uv$ in $G_c$. Since $l_{\cos}$ is the lower bound of the shortest path in $M_c$, we have
\[ d_{uv} = (1 + \epsilon)L_{u'v'} \leq (1 + \epsilon)l_{\gamma_{u'v'}} \]
where $\gamma_{u'v'}$ is the shortest path between $u'$ and $v'$, and $l_{\gamma_{u'v'}}$ is the length of $\gamma_{u'v'}$.

Also by Lemma 3, we have $U_{u'v'} = 2\pi r + 2r + l_{\cos}$, such that
\[ U_{u'v'} \leq d_{uv} \]

\[ \iff 2\pi r + 2r + d_{av}/(1 + \epsilon) \leq d_{uv} \]

\[ \iff \epsilon \geq \frac{(2\pi r + 2r)}{d_{uv} - (2\pi r + 2r)} \]

If $\epsilon \geq \frac{(2\pi r + 2r)}{d_{uv} - (2\pi r + 2r)}$, the following inequality is valid.
\[ l_{\gamma_{u'v'}} \leq U_{u'v'} \leq d_{uv} \leq (1 + \epsilon)L_{u'v'} \leq (1 + \epsilon)l_{\gamma_{u'v'}} \]

We take
\[ \epsilon = \frac{(2\pi r + 3r)}{d_{uv} - (2\pi r + 2r)} \geq \frac{(2\pi r + 2r)}{d_{uv} - (2\pi r + 2r)} \]

Since $r$ is arbitrarily close to 0 and $r << d_{uv}$, $\epsilon$ is arbitrarily close to 0.

Therefore, there exists some constant $\epsilon > 0$ arbitrarily close to 0 satisfying the following inequality.
\[ l_{\gamma_{u'v'}} \leq U_{u'v'} \leq d_{uv} \leq (1 + \epsilon)L_{u'v'} \leq (1 + \epsilon)l_{\gamma_{u'v'}} \]

The distortion of the embedding is
\[ \frac{d_{uv}}{L_{\gamma_{u'v'}}} \leq \frac{(1 + \epsilon)L_{u'v'}}{L_{u'v'}} = 1 + \epsilon \]

The distortion of embedding $G_c$ into the host Riemannian manifold $M_c$ is $(1 + \epsilon)$, where $\epsilon > 0$ and is arbitrarily close to 0. Therefore, the distortion is arbitrarily close to 1. \qed
4.3.4 Extension to the General Cases

**Theorem 10.** (Chrobak *et al.* 1996 [7]) All graphs have 3-D straight-line crossing-free drawings.

**Theorem 11.** For any undirected graph $G = (V, E)$, one can find an embedding of $G$ into some 2-D Riemannian manifold $M$ with a distortion arbitrarily close to 1.

*Proof.* Consider an arbitrary undirected graph $G = (V, E)$. To embed the graph $G$ into a Riemannian manifold $M$, we first construct $M_c$ for each pair of neighbor vertices and their in-between edges. The embedded Riemannian manifold corresponding to $i^{th}$ pair of vertices is denoted by $M_{c_i}$. Two vertices $u_i \in V$ and $v_i \in V$ are modeled as two arbitrary points $u'_i$ and $v'_i$ in their corresponding partial spheres. The shortest path between $u'_i$ and $v'_i$ is denoted by $\gamma_i$. $l_{\gamma_i}$ represents its length. The lower bound of $l_{\gamma_i}$ is denoted by $L_i$. Similarly, the upper bound of $l_{\gamma_i}$ is denoted by $U_i$. We can combine multiple $M_{c_i}$ together in 3-D space into a manifold $M$ through merging sphere portions that correspond to the same vertex. During this process, the mapped points in the spheres can be adjusted to the same point in a merged sphere since nodes are arbitrarily mapped in the spheres, and each distortion is still arbitrarily close to 1 after adjusting positions of the mapped points. While not all graphs have a planar embedding in a 2-D plane, Theorem 10 states that any graph has an embedding in 3-D space, such that no two edges intersect except at their end points. Theorem 10 and the constraint that the amplitude of Cosine curve is much smaller than the radius of spheres guarantee no Cosine portions overlap in $M$. By Definition 4 the manifold $M$ from the above combination results in Riemannian manifold since the smoothness of the surface and the smoothness of the metric tensor in the tangent space are sustained at the connections. The metric tensors are 2-D everywhere in its tangent space, thus the manifold $M$ is a 2-D Riemannian manifold.

For any two vertices $p$ and $q$ in $G$, they are modeled as two arbitrary points $p'$ and $q'$ in their corresponding partial spheres. The length of $i^{th}$ edge in $G$ is denoted by $d_i$. The
length of the shortest path between two vertices \( p \) and \( q \) is denoted by \( d_{pq} \). The shortest path between \( p' \) and \( q' \) in \( M \) is denoted by \( \gamma_{p'q'} \). \( l_{\gamma_{p'q'}} \) represents its length. The lower bound of \( l_{\gamma_{p'q'}} \) is denoted by \( L_{p'q'} \), and the upper bound of \( l_{\gamma_{p'q'}} \) is denoted by \( U_{p'q'} \). Recall that for each \( M_{c_i} \), the length of the Cosine curve \( l_{\cos_i} \) dominates the length of the shortest path, \( l_{\gamma_i} \). For the lengths of the shortest path portions in \( M_{r_1}, M_{r_2}, M_{i_1} \) and \( M_{i_2} \), they are arbitrarily close to 0, thus much smaller than any \( d_i \) or the differences between two distinct \( d_i \). We can find a lower bound for the length of the shortest path between \( p' \) and \( q' \) in \( M \) to be \( \sum_{p'q'} l_{\cos_i} \). The shortest path passes the same points corresponding to passed vertices in the shortest path from \( p \) to \( q \) in \( G \).

For the lower bound \( L_{p'q'} \), we have

\[
L_{p'q'} = \sum_{p'q'} l_{\cos_i} \leq l_{\gamma_{p'q'}}
\]

We have an upper bound \( U_{p'q'} = \sum_{p'q'} l_{\cos} + \delta_{p'q'} \geq l_{\gamma_{p'q'}} \) where \( \delta_{p'q'} \) is the summation of the lengths of the shortest paths in \( M_{i_1} \) and \( M_{r_1} \). We have shown that \( \delta_{p'q'} \) is arbitrarily close to 0 and \( \delta_{p'q'} << \sum_{p'q'} l_{\cos} \). Since \( \sum_{p'q'} l_{\cos} \) can be controlled by frequencies of the Cosine curves, we set \( L_{p'q'} = \sum_{p'q'} l_{\cos} = d_{pq}/(1 + \epsilon_{pq}) \), that is, \( d_{pq} = (1 + \epsilon_{pq})L_{p'q'} \) where \( \epsilon_{pq} \) is a positive number. We have

\[
U_{p'q'} = \sum_{p'q'} l_{\cos} + \delta_{p'q'} = d_{pq}/(1 + \epsilon_{pq}) + \delta_{p'q'}
\]

Thus

\[
U_{p'q'} \leq d_{pq}
\]

\[\iff \sum_{p'q'} l_{\cos} + \delta_{p'q'} \leq d_{pq} \]

\[\iff d_{pq}/(1 + \epsilon_{pq}) + \delta_{p'q'} \leq d_{pq} \]
\[\iff \epsilon_{pq} \geq \frac{\delta_{p'q'}}{d_{pq} - \delta_{p'q'}}\]

We can take \(\epsilon_{pq} = \frac{2\delta_{p'q'}}{d_{pq} - \delta_{p'q'}} \geq \frac{\delta_{p'q'}}{d_{pq} - \delta_{p'q'}}\), such that

\[U_{p'q'} \leq d_{pq}\]

Such \(\epsilon_{pq}\) satisfies the above inequality and is arbitrarily close to 0. Therefore, we have the following inequality

\[L_{p'q'} \leq L_{\gamma_{p'q'}} \leq U_{p'q'} \leq d_{pq} = (1 + \epsilon_{pq})L_{p'q'} \leq (1 + \epsilon_{pq})l_{\gamma_{p'q'}}\]

that is,

\[l_{\gamma_{p'q'}} \leq d_{pq} \leq (1 + \epsilon_{pq})l_{\gamma_{p'q'}}\]

The distortion of the embedding is

\[\frac{d_{pq}}{l_{\gamma_{p'q'}}} \leq \frac{(1 + \epsilon_{pq})L_{p'q'}}{L_{p'q'}} = 1 + \epsilon_{pq}\]

where \(\epsilon_{pq}\) is arbitrarily close to 0.

Therefore, for any undirected graph \(G\), one can find an embedding of \(G\) into some 2-D Riemannian manifold \(M\) with a distortion arbitrarily close to 1.

\[\square\]

4.4 A Riemannian Manifold Approach for Multiple-unicast Network Coding

**Theorem 12.** If the multiple-unicast conjecture is true in 2-D Riemannian manifolds, then it is true in undirected networks.

**Proof.** We prove this by way of contradiction.

Consider an undirected graph \(G = (V, E)\). By Table. 4.1, the multiple-unicast conjecture in throughput domain can be translated into cost domain. \(G\) can be converted into a cost-weighted undirected network \((G, \omega)\). \(d_i\) is the length of the shortest path between the
terminals of $i^{th}$ unicast pair, $s_i$ and $t_i$, in $G$. Each pair of unicast is equipped with a desired throughput $r_i$. The minimum cost under routing is $\sum_i (d_i r_i)$. Each link $e$ has a cost $\omega(e)$. Let $f$ be an underlying flow vector for a network coding solution. Thus the corresponding total cost under network coding is $\sum_e (\omega(e) f(e))$.

Assume the multiple-unicast conjecture fails in $(G, \omega)$. Thus in cost domain, there exists a network coding solution with its underlying flow vector $f$ such that

$$\sum_e (\omega(e) f(e)) < \sum_i (d_i r_i)$$

Furthermore, according to Theorem 11, $G$ can be embedded into a 2-D Riemannian manifold $M$ with a distortion arbitrarily close to 1, that is,

$$l_{\gamma_i} \leq d_i \leq (1 + \epsilon_i) l_{\gamma_i}$$

where $\epsilon_i$ is arbitrarily close to 0. The corresponding desired throughput $r_i$ remains the same. Thus

$$\sum_i (l_{\gamma_i} r_i) \leq \sum_i (d_i r_i) \leq (1 + \epsilon_{max}) \sum_i (l_{\gamma_i} r_i)$$

At the meantime, we keep the same flow for each link in the Riemannian manifold $M$. The cost $\omega_M(e)$ in $M$ could be reduced as the distance reduced ($l_{\gamma_i} \leq d_i$). Thus

$$\sum_e (\omega_M(e) f(e)) \leq \sum_e (\omega(e) f(e))$$

We have in the 2-D Riemannian manifold $M$

$$\sum_e (\omega_M(e) f(e)) \leq \sum_e (\omega(e) f(e)) < \sum_i (d_i r_i) \leq (1 + \epsilon_{max}) \sum_i (l_{\gamma_i} r_i)$$

where $\epsilon_{max}$ is arbitrarily close to 0.

We conclude $\sum_e (\omega(e) f(e)) < \sum_i (l_{\gamma_i} r_i)$ as $\epsilon_{max}$ is arbitrarily close to 0. This contradicts the assumption that multiple-unicast conjecture is true in 2-D Riemannian manifolds, i.e.,
\[ \sum_e (\omega(e)f(e)) \geq \sum_i (l_{\gamma}r_i). \]

Therefore, if the multiple-unicast conjecture is true in 2-D Riemannian manifolds, then it is true in undirected networks.

\[ \square \]

**Table 4.1:** The multiple-unicast conjecture from throughput to cost domain.

<table>
<thead>
<tr>
<th>The multiple-unicast Conjecture [18]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Throughput Domain:</strong> For ( k ) independent unicast sessions ( in ) a capacitated undirected network ( (G,c) ), a throughput vector ( r ) is feasible with network coding if and only if it is feasible with routing.</td>
</tr>
<tr>
<td><strong>Cost Domain:</strong> Let ( f ) be the underlying flow vector of a network coding solution for ( k ) independent unicast sessions with throughput vector ( r ) ( in ) a cost-weighted undirected network ( (G,w) ). Then ( \sum_e (\omega(e)f(e)) \geq \sum_i (d_ir_i) ).</td>
</tr>
</tbody>
</table>

**Corollary 1.** In the multiple-unicast setting, if the coding advantage is upper bounded by a constant in 2-D Riemannian manifolds, then it is also upper bounded by a constant in undirected networks.

**Proof.** We prove this by way of contradiction.

Assume the coding advantage is not upper bounded by a constant \( h \) in \( (G,\omega) \), where \( h \) can be any constant. Thus in cost domain, there exists a network coding solution with its underlying flow vector \( f \) such that

\[ h \sum_e (\omega(e)f(e)) < \sum_i (d_ir_i) \]

By Theorem [11] \( G \) can be embedded into a 2-D Riemannian manifold with a distortion arbitrarily close to 1. We have

\[ h \sum_e (\omega_M(e)f(e)) \leq h \sum_e (\omega(e)f(e)) < \sum_i (d_ir_i) \leq (1 + \epsilon_{\max}) \sum_i (l_{\gamma}r_i) \]

where \( \epsilon_{\max} \) is arbitrarily close to 0.
We conclude $h \sum_e (\omega(e) M f(e)) < \sum_i (l_i r_i)$ as $\epsilon_{max}$ is arbitrarily close to 0. This contradicts the assumption that the coding advantage is upper bounded by a constant in 2-D Riemannian manifolds, i.e., $h \sum_e (\omega(e) M f(e)) \geq \sum_i (l_i r_i)$. Therefore, in the multiple-unicast setting, if the coding advantage is upper bounded by a constant in 2-D Riemannian manifolds, then it is also upper bounded by a constant in undirected networks.

\textbf{Theorem 13.} (Braverman et al. 2016 \cite{6}) Given a multiple-unicast undirected network $G = (V, E)$ with coding advantage $1 + \zeta$, where $\zeta$ is a positive constant, we can construct a sequence of networks $G_i = (V_i, E_i)$ with coding advantage at least $(1 + \zeta)^2$ and size at most $(3h_m)^{(4h_1)^{2i+1}}$ where $h_m$ and $h_1$ are absolute constants.

\textbf{Corollary 2.} In the multiple-unicast setting, if the coding advantage is upper bounded by a constant in 2-D Riemannian manifolds, then the multiple-unicast conjecture is true in undirected networks.

\textbf{Proof.} Theorem 13 states that if there exists a multiple-unicast undirected network with coding advantage larger than 1, then there exists a multiple-unicast undirected network with coding advantage approach infinity. An equivalent statement for Theorem 13 is that if there is a constant upper bound of the coding advantage for multiple-unicast network coding in undirected networks, then the multiple-unicast conjecture is true. In other words, the conjecture is equivalent to claiming that there is a constant upper bound of the coding advantage. Given Corollary 1, we can conclude that if the coding advantage is upper bounded by a constant in 2-D Riemannian manifolds, then the coding advantage is also upper bounded by a constant in undirected networks, which implies the multiple-unicast conjecture in undirected networks. \hfill \Box
Chapter 5

The Equivalence between Routing and Network Coding in Constant Curvature Manifolds

In this chapter, we continue exploring the multiple-unicast conjecture in some of the most basic cases of 2-D Riemannian manifolds. Riemannian manifolds of constant curvature are on one hand the simplest Riemannian manifolds and in the other hand sufficiently complicated enough to present a fascinating study [22]. The conjecture has been verified on Euclidean spaces which are Riemannian manifolds with constant 0 curvature [32]. We aim to discover some new scenarios where the conjecture is true. Therefore, we study three types of constant curvature 2-D Riemannian manifolds: positive constant, constant 0 and negative constant curvature Riemannian manifolds. We show that network coding does not have advantage over routing in some of these elementary Riemannian manifolds.

5.1 The MUNC Conjecture in Positive Constant Curvature Manifolds

According to Definition 11, a sphere, $S^2$, is a positive constant curvature 2-D manifold. Theorem 14 characterizes the shortest paths in a sphere.

**Theorem 14.** (Weisstein, 2016 [28]) The shortest path between two points in a sphere is a segment of a great circle along the surface of the sphere.

The general approach of our proof for the multiple-unicast network coding conjecture in a sphere is to project the cost domain solution in a sphere as the combinations in unit circles. Theorem 15 shows that the multiple-unicast conjecture is true in unit circles.

**Theorem 15.** Given $k$ independent unicast sessions in a circle $S^1$, let $f$ be the underlying flow vector of a network coding solution achieving a rate vector $r$. Then $\sum_{e} (f(e) \| \hat{e} \|) \geq \ldots$
\[ \sum_i (d_{s_i t_i} r_i). \]

**Proof.** In a circle \( S^1 \) of radius \( r \), we have a line passing through the center point which intersects \( S^1 \) at two points \( p_0 \) and \( p_1 \) as shown in Fig. 5.1. \( p_0 \) and \( p_1 \) form a cut of the network. The total amount of flow passing \( p_0 \) and \( p_1 \) in both directions is at least the throughput requirement of terminal pairs separated by the cut.

![Figure 5.1: Three unicast sessions in a circle \( S^1 \). Total flow crossing two points \( p_o \) and \( p_1 \), \( f_{p_{\theta,0}} + f_{p_{\theta,1}} \) is lower bounded by Demand from the half circle anticlockwise to \( p_{\theta,0} \) and the half circle clockwise to \( p_{\theta,0} = r_2 \).](image)

We rotate the line clockwise by \( \theta \), where \( \theta \in [0, \pi] \). The rotated line intersects \( S^1 \) at \( p_{\theta,0} \) and \( p_{\theta,1} \). Let \( l \) denote the length of the minor arc \( p_{\theta,0}p_{\theta,0} \). Thus, \( l \) changes from 0 to \( \pi r \). Let \( f_{p_{\theta,0}} + f_{p_{\theta,1}} \) be the total amount of flow crossing \( p_{\theta,0} \) and \( p_{\theta,1} \) in both directions. \( p_{\theta,0} \) and \( p_{\theta,1} \) forms a cut. Thus \( f_{p_{\theta,0}} + f_{p_{\theta,1}} \) is lower bounded by demand between the half circle anticlockwise to \( p_{\theta,0} \), \( p_{\theta,0}p_{\theta,1} \), and the half circle clockwise to \( p_{\theta,0} \), \( p_{\theta,0}p_{\theta,1} \), denoted by \( \text{Demand}(p_{\theta,1}p_{\theta,0}, p_{\theta,0}p_{\theta,1}) \). Let \( d_{s_i t_i} \) denote the length of shortest path between \( s_i \) and \( t_i \). We integrate both quantities for \( l \) from 0 to \( \pi r \). We have

\[
\int_0^{\pi r} (f_{p_{\theta,0}} + f_{p_{\theta,1}}) dl \geq \int_0^{\pi r} \text{Demand}(p_{\theta,1}p_{\theta,0}, p_{\theta,0}p_{\theta,1}) dl
\]
= \sum_i (d_{s_it_i} r_i)

Furthermore, note that \(\sum_{e} (||\hat{e}|| f(\hat{e})) = \int_0^{\pi \rho} (f_{\rho \alpha} + f_{\rho \beta}) dl\). We conclude that \(\sum_{e} (||\hat{e}|| f(\hat{e})) \geq \sum_i (d_{s_it_i} r_i)\). Therefore, the multiple-unicast conjecture is true in \(S^1\). \(\square\)

**Theorem 16.** The multiple-unicast network coding conjecture is true in 2-D spheres, which have a positive constant curvature.

**Proof.** We prove the theorem by way of contradiction.

The proof has two major steps. (i) We show the network cost in \(S^2\) is a linear combination of network costs in the great circles. (ii) The multiple-unicast conjecture is true in great circles.

We have a general undirected graph \(G = (V, E)\) in a sphere. Without losing generality, we set the radius of our 2-D sphere \(S^2\) as 1. An edge in \(G\) is denoted by \(\hat{e}\) and its length by \(||\hat{e}||\). The flow in an edge \(\hat{e}\) is denoted by \(f(\hat{e})\), and the shortest path between the \(i^{th}\) pair of source and receiver is denoted by \(d_{s_it_i}\) with desired throughput \(r_i\).

Given the multiple-unicast network coding problem in \(S^2\), in the cost domain, we assume there exists a network coding solution with its underlying flow vector \(f\) such that

\[
\sum_{\hat{e}} (f(\hat{e})||\hat{e}||) < \sum_i (d_{s_it_i} r_i)
\]

We construct \(k\) pairs of unicast instance \(S^1\) and their network coding solutions by projecting arcs from \(S^2\) to \(S^1\). Our goal is to show that there exists an \(S^1\) sub-space in the \(S^2\), where the projection satisfies

\[
\sum_{\hat{e}} (f(\hat{e})||\hat{e}||) < \sum_i (d_{s_it_i} r_i)
\]

According to Theorem 14, the shortest path between two points in \(S^2\) is a part of a great circle, thus \(\hat{e}\) and \(s_it_i\) are arcs of great circles in \(S^2\). The maximum length of the shortest
path in $S^2$ equals to the length of a half great circle, which is $\pi$. Let $p$ be any point in $S^2$. A point $p$ is associated with a unique great circle $\hat{c}$, which is contained in the plane that is orthogonal to the line connecting $p$ and the sphere center. Let $\hat{1}$ be a unit arc in some great circle. We project $\hat{1}$ to every possible great circle. By such projection, arcs in $S^2$ are projected into multiple $S^1$ spaces. Each projection is associated with a point $p$ in $S^2$. We can enumerate all lengths of possible projections in $S^1$ by traversing all points in $S^2$. We use $\Phi$ to denote the surface of $S^2$. Fig. 5.2 shows an example of the projection. The point $P$ locates at the north pole $(0, 0, 1)$. An arc as a part of some great circle is projected into the great circle associated with the north pole $P$, which is the equator. From $P$, two great circle intersect with the end points of the arc and the equator. The smaller arc in the equator from the intersection is the projection of the arc. By traversing $P$ to every points in the surface, we can enumerates the lengths of projections of some arc into every possible great circle in the sphere.

![Figure 5.2: An illustration of the projection in a sphere.](image)

We show that the enumeration of the lengths of projections is linearly dependent on the length of an arc, thus without losing generality, we can simply project an arc of unit length, $\hat{1}$. For any arc, we can divide it into small segments of arcs of the same length. We observe
that when integrating the lengths of projections of the arc, the position or the orientation of the arc does not affect the results since we project it into all possible great circles. Different situations from various positions and orientations can be transformed into equivalence by simply rotating the sphere. Therefore, all small segments of arcs with the same length can be projected as indicated above, resulting in the same integration. Besides, there are no overlaps when projecting these arc segments into the the same great circle, the integration of the lengths of projections in all possible great circles for an arc is the summation of the total lengths of projections in all possible great circles of its arc segments. Thus the enumeration of the lengths of projections for an arc equals the number of arc segments multiplying the integration of lengths of projections of each arc segment. The lengths of its arc segments are the same so as the enumerations of the lengths of projections of the segments. Therefore, the integration of the lengths of projections for an arc in a sphere is linearly dependent on its length.

The integration over $S^2$ for all lengths of projections of $f$ is

$$
\iint_{S^2} \sum_{\hat{e}} f(\hat{e}) (\hat{e} \cdot \hat{c}) d\Phi = \sum_{\hat{e}} \iint_{S^2} f(\hat{e}) (\hat{e} \cdot \hat{c}) d\Phi
$$

$$
= \sum_{\hat{e}} \iint_{S^2} f(\hat{e}) ||\hat{c}|| (\hat{1} \cdot \hat{c}) d\Phi
$$

$$
= \sum_{\hat{e}} (f(\hat{e}) ||\hat{c}||) \iint_{S^2} (\hat{1} \cdot \hat{c}) d\Phi
$$

We illustrate the integration of the lengths of projections when $p$ traverses the complete great circle in $x - z$ plane in Fig. 5.3, where the projected unit arc locates in $x - y$ plane. The integration is

$$
4 \int_0^{\pi} 1 \cdot |\tan^{-1} \left( -\frac{\tan(1)}{\cos(x)} \right) | \, dx
$$

Further, we can regard the complete enumeration as the size of an area (in Fig. 5.3) integrated along the rotation by the $z$ axis. When extending to the complete integration, we apply the same calculation, but project with the integration along $y$ axis instead of unit length. The
integration thus is

\[ 4 \int_{0}^{\frac{\pi}{2}} \left| 4 \int_{0}^{\frac{\pi}{2}} \tan^{-1} \left( -\frac{\tan(1)}{\cos(x)} \right) \cdot \tan^{-1} \left( -\frac{\tan(1)}{\cos(x)} \right) \right| \, dx \]

The result is a constant, which is

\[
(\log_{2} \left( \tan^{2} \left( \frac{1}{2} \right) \right) - 4\log_{2} \left( \tan \left( \frac{1}{2} \right) \right) -
4 \log \left( \cot \left( \frac{1}{2} \right) \right) \coth^{-1} \left( \cot \left( \frac{1}{2} \right) \right)^{2}
\]

In Fig. 5.3, the area stands for the integrated lengths of projection of an arc \( \hat{e} \) in all possible great circles in \( y \) axis regardless of the orientation of \( \hat{e} \). By further enumerating through all possible great circles, the integration is about 56.2374.

![Figure 5.3: Integration of Projections over great circle that are rotations of the equator along with \( y \) axis between 0 and \( 2\pi \).](image)

The integration over \( S^{2} \) for all lengths of projections of \( \{s_{i}t_{i}r_{i} \mid i = 1, \ldots, k \} \) is

\[
\int_{\Phi} \sum_{i} (s_{i}t_{i} \cdot \hat{c}) r_{i} d\Phi = \sum_{i} \int_{\Phi} (s_{i}t_{i} \cdot \hat{c}) r_{i} d\Phi
\]

\[
= \sum_{i} \int_{\Phi} d_{s_{i}t_{i}} (1 \cdot \hat{c}) r_{i} d\Phi
\]

\[
= \sum_{i} d_{s_{i}t_{i}} r_{i} \int_{\Phi} (1 \cdot \hat{c}) d\Phi
\]
Since $\sum_{\hat{e}} f(\hat{e})||\hat{e}|| < \sum_i (d_{s_i t_i} r_i)$ by assumption, we claim that

$$\int_{\Phi} \sum_{\hat{e}} (f(\hat{e}) (\hat{e} \cdot \hat{c})) d\Phi < \int_{\Phi} \sum_i (s_i t_i \cdot \hat{c}) r_i d\Phi$$

There must exist a particular projection to $\hat{c}^*$, for which

$$\sum_{\hat{e}} f(\hat{e}) (\hat{e} \cdot \hat{c}^*) d\Phi < \sum_i (s_i t_i \cdot \hat{c}^*) r_i d\Phi \tag{5.1}$$

However, according to Theorem 15, the multiple unicast network coding conjecture is true in $S^1$, which indicates

$$\sum_{\hat{e}} f(\hat{e})||\hat{e}|| d\Phi \geq \sum_i ||s_i t_i \cdot \hat{c}|| r_i d\Phi$$

for any $\hat{c}$. It contradicts Eq. 5.1.

Therefore, in $S^2$, we have $\sum_{\hat{e}} f(\hat{e})||\hat{e}|| \geq \sum_i (d_{s_i t_i} r_i)$ for all coding solutions. We conclude that the multiple-unicast network coding conjecture is true in $S^2$.

5.2 The MUNC Conjecture in Zero Constant Curvature Manifolds

The case of Euclidean space as the most common case of zero constant curvature Riemannian manifolds was proved by Xiahou et al. 32. In this section, we intend to show that in some other cases of zero constant curvature manifolds, the multiple-unicast network coding conjecture is also true. We begin with the case of cylindrical tubes. According to Grady and Polimeni 12, the cylindrical tube is 2-D Riemannian manifold with a zero Gaussian curvature.

**Theorem 17.** The multiple-unicast network coding conjecture is true in cylindrical tubes, which have a constant curvature of 0.

**Proof.** We prove the theorem by way of contradiction.
The proof has two major steps. (i) We show the network cost in the cylindrical tube is a linear combination of network costs in loops. (ii) The multiple-unicast conjecture is true in loops.

The shortest path in a cylindrical tube is a curve that would be a straight line the manifold is unrolled into a flat plane.

We have a general undirected graph $G = (V, E)$ in a cylindrical tube $T$. Without losing generality, we set the radius of the cylindrical tube as 1 and the vertical length as infinity. An edge in $G$ is denoted by $\hat{e}$ and its length by $||\hat{e}||$. The flow in an edge $\hat{e}$ is denoted by $f(\hat{e})$, and the shortest path between the $i$th pair of source and receiver is denoted by $d_{s_i t_i}$ with desired throughput $r_i$.

Given the multiple-unicast network coding problem in a cylindrical tube $T$, we assume there exists a network coding solution with its underlying flow vector $f$ such that

$$\sum_{\hat{e}} (f(\hat{e})||\hat{e}||) < \sum_i (d_{s_i t_i} r_i)$$

We process the $k$ pairs of unicast instance and its network coding solution by projecting the shortest paths from $T$. Our goal is to show that there exists a 1-D loop sub-space in the $T$, in which the projection satisfies

$$\sum_{\hat{e}} (f(\hat{e})||\hat{e}||^1) < \sum_i (d_{s_i t_i} r_i)$$

Fig. 5.4 shows a cross section of $\theta$ degree. The boundary of the $\theta$ degree cross section can be unwrapped into piece-wised lines line. Let $\hat{1}$ be a unit geodesic in the cylindrical tube. We project $\hat{1}$ to the boundary of the cross section with degree $\theta$. We can enumerate all lengths of projections when $\theta$ varies from 0 to $\pi$. The expanded view of the cross section is illustrated in Fig. 5.5. First we show that the enumeration is independent of the position and rotation of an geodesic. In other words, any unit geodesic in the cylindrical tube has the same constant accumulation of the lengths of projections into all boundaries of the cross

53
sections.

The enumeration is independent of the movements of the unit geodesic along the $z$ axis. We pay attention to the independence of rotations along the $x-y$ plane. For a unit geodesic of rotation $\alpha$ degree, the length of projection into a cross section of degree $\theta$ is $\cos(\theta - \alpha)$, when the unit geodesic resides solely in the $x > 0$ portion. The length of projection into a cross section of degree $\theta$ is $\sin(\theta - \alpha)$, when the unit geodesic resides solely in the $x < 0$ portion. Fig. 5.4 illustrates the projection.

The integration of the lengths of projections of the unit geodesic solely in $x > 0$ portion when $\theta$ traverses from 0 to $\pi$ is

$$\int_0^\pi |\cos(\theta - \alpha)| \, d\theta = 2$$

which is a constant. The result can also be observed, shown as $l_1$ in Fig. 5.5

Similarly, the integration of the lengths of projections of the unit geodesic solely in $x < 0$
portion, shown as $l_2$ in Fig. 5.5 when $\theta$ traverses from 0 to $\pi$ is

$$\int_{0}^{\pi} |\sin(\theta - \alpha)| \ d\theta = 2$$

which is a constant as well.

As to the situation when the unit geodesic intersects with the axis of $x = 0$, we illustrate as $l_3$ in Fig. 5.5. The length of projection is a linear combination (the summation of the factors is 1.) of the length of projection when $x > 0$ and $x < 0$ as shown in Fig. 5.6. Since both integrations of $\theta$ traversing from 0 to $\pi$ are a constant 2, the integration of the lengths of projection of the unit geodesic intersecting with $x = 0$ is also a constant 2.

Thus the enumeration of the lengths of our projection is independent of the position and rotation of the geodesic. At the meantime, the enumeration is linearly dependent on the length of the geodesic.

Let $\hat{e} \cdot \hat{c}$ denote the length of projection to the cross sections. The integration over $T$ for all lengths of projections of $f$ is

$$\oint_{\Phi} \sum_{\hat{e}} f(\hat{e})(\hat{e} \cdot \hat{c})d\Phi = \sum_{\hat{e}} \oint_{\Phi} f(\hat{e})(\hat{e} \cdot \hat{c})d\Phi$$
Figure 5.6: The integration of a unit geodesic in the cylindrical tube. The integration equals the area of the shaded region of length $\pi$. The area is a constant independent of the value of $\alpha$.

\[
\begin{align*}
= & \sum_i \iiint_{\Phi} f(\hat{\mathbf{e}})||\hat{\mathbf{e}}|| (\hat{\mathbf{1}} \cdot \hat{\mathbf{e}}) d\Phi \\
= & \sum_i (f(\hat{\mathbf{e}})||\hat{\mathbf{e}}||) \iiint_{\Phi} (\hat{\mathbf{1}} \cdot \hat{\mathbf{e}}) d\Phi
\end{align*}
\]

The integration over $T$ for all lengths of projections of \{s_i t_i r_i | i = 1, ..., k\} is

\[
\begin{align*}
\iiint_{\Phi} \sum_i (s_i t_i \cdot \hat{\mathbf{c}}) r_i d\Phi = & \sum_i \iiint_{\Phi} (s_i t_i \cdot \hat{\mathbf{c}}) r_i d\Phi \\
= & \sum_i \iiint_{\Phi} d_{s_i t_i} (\hat{\mathbf{1}} \cdot \hat{\mathbf{c}}) r_i d\Phi \\
= & \sum_i d_{s_i t_i} r_i \iiint_{\Phi} (\hat{\mathbf{1}} \cdot \hat{\mathbf{c}}) d\Phi
\end{align*}
\]

Since $\iiint_{\Phi} (\hat{\mathbf{1}} \cdot \hat{\mathbf{c}})$ is a constant and $\sum_{\hat{\mathbf{e}}} f(\hat{\mathbf{e}})||\hat{\mathbf{e}}|| < \sum_i (d_{s_i t_i} r_i)$ by assumption, we claim that

\[
\begin{align*}
\iiint_{\Phi} \sum_{\hat{\mathbf{e}}} (f(\hat{\mathbf{e}})(\hat{\mathbf{e}} \cdot \hat{\mathbf{c}})) d\Phi < & \iiint_{\Phi} \sum_i (s_i t_i \cdot \hat{\mathbf{c}}) r_i d\Phi
\end{align*}
\]

There must exists a particular projection to $\hat{\mathbf{c}}*$, for which

\[
\sum_{\hat{\mathbf{e}}} (f(\hat{\mathbf{e}})(\hat{\mathbf{e}} \cdot \hat{\mathbf{c}}*)) d\Phi < \sum_i (s_i t_i \cdot \hat{\mathbf{c}}*) r_i d\Phi \tag{5.2}
\]
However, according to Theorem 15, the multiple-unicast conjecture is true in $S^1$. The boundary of the cross section is a loop which is isometric to $S^1$. It implies that the multiple-unicast conjecture is also valid in the boundary of the cross section. We have

$$\sum_{\hat{c}} (f(\hat{c}) ||\hat{c}||) d\Phi \geq \sum_i d_{s_i t_i} \cdot c_i d\Phi$$

for any $\hat{c}$. That contradicts Eq. 5.2.

Therefore, in the cylindrical tube, we have $\sum_{\hat{c}} (f(\hat{c}) ||\hat{c}||) \geq \sum_i (d_{s_i t_i} r_i)$ for all coding solutions. We conclude that the multiple-unicast network conjecture is true in cylindrical tubes.

We can extend the result above. Although prism tubes are discrete manifolds, not Riemannian manifolds, they can all be unrolled into flat planes, and the shortest paths in them are straight lines in their unrolled planes. In other words, prism tubes can be isometrically embedded into cylindrical tubes. They share the same property we use to prove the multiple-unicast conjecture in cylindrical tube. Therefore, similarly, the multiple-unicast network conjecture is true in all prism surfaces.

Flat torus is another case of everywhere zero constant curvature manifold. However, transforming cylindrical tube into flat torus stretches the manifold. Flat torus can not be isometric embedded into cylindrical tube or Euclidean space. It would require a different approach to prove multiple-unicast conjecture in them. We will leave this problem as future work.

5.3 The MUNC Conjecture in Negative Constant Curvature Manifolds

The study of negative constant curvature manifolds is known as hyperbolic geometry. The hyperbolic plane is abstract to study directly, so there are four models commonly used for hyperbolic geometry: the Klein model, the Poincare disk model, the Poincare half-plane
model, and the Lorentz or hyperboloid model \[30\]. These models define a hyperbolic plane which satisfies the axioms of hyperbolic geometry. All these models are extendable to high dimensions. Below we review some theoretical foundations of hyperbolic geometry.

To verify the multiple-unicast conjecture in hyperbolic geometry, we construct our model based on one of the hyperbolic models. By comparing among the properties of the models, the Poincare disk model is chosen, for its geometric structure built upon a circle, as shown in Fig. 5.7.

![Figure 5.7: An illustration of the Poincare disk model. A shortest path is called a hyperbolic line. Given two distinct points \( p \) and \( q \) inside the disk, the unique hyperbolic line connecting them intersects the boundary at two ideal points where the hyperbolic line is perpendicular to the boundary.](image)

Given a hyperbolic unit disk, namely the Poincare disk, for any two distinct points \( p \) and \( q \), we are able to calculate the circle passing point \( p(u_1, u_2) \) and point \( q(v_1, v_2) \) \[31\], which is

\[
x^2 + y^2 + \frac{u_2(v_1^2 + v_2^2) - v_2(u_1^2 + u_2^2) + u_2 - v_2}{u_1v_2 - u_2v_1} x + \frac{v_1(u_1^2 + u_2^2) - u_1(v_1^2 + v_2^2) + v_1 - u_1}{u_1v_2 - u_2v_1} y + 1 = 0
\]

The hyperbolic distance \( d_{pq} \) between two points in the unit disk can be evaluated by integrating \( ds \), which is
\[ ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2} \]

The central angle \( \beta \) of \( \widehat{pq} \) is \( \frac{d_{pq}}{r} \).

Unfortunately, the conventional approach for proving the multiple-unicast network coding conjecture in the previous sections does not apply in the hyperbolic plane. So far we do not have a proof of the conjecture for negative constant curvature 2-D Riemannian manifolds. We leave the proof as future work.

5.4 Approach Revisited with Information Inequalities

In the previous sections for proving the multiple-unicast network coding conjecture in constant curvature manifolds, we adopt the methodology from the work by Xiahou et al. [32]. In this section, we discuss some improvements of the approach by introducing information inequalities. We first provide the definition of cut-set bound and information inequality.

Without losing generality, in this section, we consider an undirected graph \( G(V, E) \) of unit edge length. An example of a cut is shown in Fig. 5.8

**Definition 15.** (Weisstein, 2016 [27]) A cut of a graph is a set of edges of a graph whose removal disconnects the graph.

We have the cut-set bound

\[ \sum_{e \in F} f_e \geq \sum_{i \in \text{Sep}(F)} H(X_i) \]

**Definition 16.** (Informational Dominance, Harvey et al. 2016, [13]) An edge set \( A \) informationally dominates edge set \( B \) if for all network coding solutions and \( k \)-tuples of messages \( x \) and \( y \), \( f_A(x) = f_A(y) \) implies \( f_B(x) = f_B(y) \).

Harvey et al. [13] first presented the concept of information dominance along with the Input-output Inequality and Crypto Inequality. Let \( \mathcal{A} \) be the set of directed links obtained
Figure 5.8: Nodes of a graph divided into two parts by a set of edges.

from $E$ by replacing each undirected edge with a pair of opposite directed links. Let $\mathcal{X} = \{X_i, i = 1, 2, \ldots, k\}$ denote the set of source messages. For a coding solution, let $X_{uv}$ be the messages transmitted in link $uv \in A$. Let $H(A)$ represent the entropy of the messages transmitted in the links in $A$ where $A$ is a set $A \in \mathcal{A} \cup \mathcal{X}$. If the edge set $B$ is dominated by the edge set $A$, we have $H(A) \geq H(B)$. The Input-output Inequality indicates the information leaving a vertex set is dominated by the information entering the set. Crypto inequality states that if a graph $G$ has a cut set, the cut set informationally dominates all the source information whose source and sink are separated. Inspired by information inequality, Yin et al. [33] formulated an alternative conjecture for multiple-unicast network coding.

**Theorem 18.** (Yin el al. 2016 [33]) The multiple-unicast network coding conjecture is equivalent to the following inequality: for an undirected network $G(V, E)$ of unit link length,

$$\forall F \subset E : \sum_{e=uv \in F} H(uv) + H(vu) \geq \sum_i d_{G/F}(s_i, t_i)H(X_i)$$

where $G/\bar{F}$ is the resulting graph obtained from $G$ by contracting all edges not in $F$.

Our current approach is based on flow inequality in the cost domain. By adopting Theorem 18, we can apply the entropy $H(X_{uv}), H(X_{vu})$ instead of the flow rates $f_{uv}$, to take advantage of the submodularity of information inequality. That is
\[ H(A) + H(B) \geq H(A \cap B) + H(A \cup B) \]

The methodology in this paper requires to project the network from a high dimensional manifold to the same type of manifolds of low dimension. Thanks to the cascade way of combining entropy terms, we can extend this aspect by allowing projecting the graph in a manifold \( M \) to a variety of manifolds of different dimensions. We can show the correctness of the conjecture in \( M \) by showing that (i) the network cost is a linear combination of the costs in these manifolds, and (ii) the multiple-unicast network coding conjecture is true in all of these Riemannian manifolds.

We summarize the refined approach in Table 5.1.
Table 5.1: Comparison before and after introducing information inequality.

<table>
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<tr>
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<th>Current Approach</th>
<th>Approach with Information Inequality</th>
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<tr>
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<td>the given Riemannian manifold</td>
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Chapter 6

Conclusion and Open Problems

6.1 Thesis Summary

This thesis begins with an introduction of network coding and the long standing open problem, the multiple-unicast conjecture.

After reviewing related studies and preliminaries, in Chapter 4, we present an approach to embed undirected graphs into a Riemannian manifold. The distortion of our embedding can be arbitrarily close to 1. We further show that if the multiple-unicast conjecture is true on 2-D Riemannian manifolds, then the conjecture is true in general undirected networks. Consequently, we might be able to prove the original multiple-unicast network coding conjecture by studying the conjecture on Riemannian manifolds.

In Chapter 5, we focus on a number of basic Riemannian manifolds with constant curvature, and prove that the multiple-unicast network coding conjecture is true in them, including the sphere (positive constant curvature) and the cylindrical tube (constant zero curvature). Our result extends to any manifold that can be isometrically embedded into sphere, cylindrical tube and flat plane. The case in negative constant manifolds is introduced but yet to be resolved. A refined approach to the original method by Xiahou et al. [32] is discussed and proposed, which combines the geometric approach with theory of information inequalities and the latest result by Xunrui et al. [33].

6.2 Future Work

In Chapter 4, we conclude that if the multiple-unicast conjecture is true in 2-D Riemannian manifolds, then the conjecture is true in general undirected networks. The ultimate goal is
to prove (or disprove) the conjecture. We discover a new approach to examine the conjecture in general scenarios. In future work, one may attempt to determine the conjecture in Riemannian manifolds and hopefully resolve the long standing open problem through this more refined geometric approach. In Chapter 5, we leave the case of multiple-unicast in flat torus and negative constant curvature manifolds as open problems.
Bibliography


